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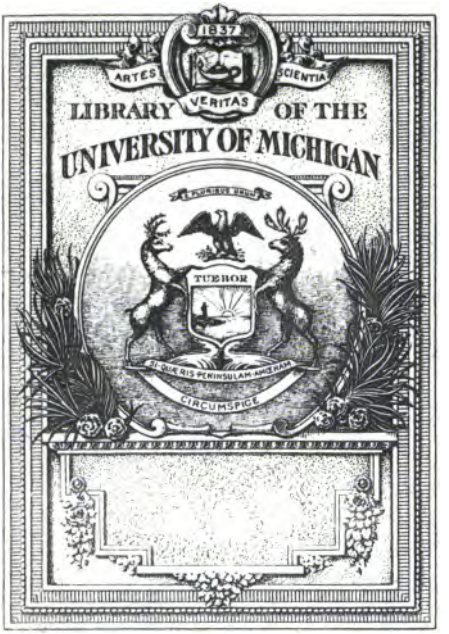

THE GIFT OF
Prof. Louis C. Karpinski

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THE TEACHING OF ARITHMETIC

THE TEACHING OF ARITHMETIC

A MANUAL FOR TEACHERS

BY

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INTRODUCTION

A STATEMENT of the aims of a book on the methods of teaching should set forth its distinctive purpose. Books on the pedagogy of arithmetic which have contributed to improvement in methods of teaching this subject may be divided into three classes. Some seek to give a general survey of the aims and methods of teaching arithmetic, concerning themselves with no details in any specific topic, and leaving it to each teacher to apply the general method to the specific problems of his grade work. Other books give the philosophy and psychology of mathematics in the hope that a clearer comprehension of the basic concepts of arithmetic by the teacher will inevitably lead to more effective methods of instruction. The third group of books devotes itself merely to the task of giving remedial devices to meet specific teaching difficulties. General methods of teaching any subject may set forth ideals of instruction without regard to possibility of attainment. Specific devices tend to establish stereotyped forms to be followed slavishly without a comprehension of the basic principles. The writer has sought to evolve a book that is eclectic in aim. The early chapters study critically the values of arithmetic, the principles governing the organization of the course of study, and the psychology underlying sound method in arithmetic. The later chapters set forth methods rather than the method of teaching each of the important branches of arithmetic. The aim of the book is, therefore, to evolve a plan of teaching which is based on approved

psychology of number, which incorporates the lessons of contemporary research in methodology and which has stood the final test of experience. This book is not a text on the subject of arithmetic, but a manual of method of teaching arithmetic. It is an outgrowth of a course of lectures given to teachers of the elementary schools. The author begs to express his indebtedness to the many teachers in whose class work he has found much that may be of practical worth in this book. He is especially grateful to Mr. Mesmin Arenwald, Instructor of Mathematics in Evander Childs High School, for many valuable suggestions and for a very careful reading of the proofs.

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THE TEACHING OF ARITHMETIC

CHAPTER I

THE VALUES OF ARITHMETIC

WHAT CAN ARITHMETIC GIVE THE CHILD?

Need of Formulating the Values of Arithmetic.—The initial problem which confronts the teacher interested in discovering the most effective methods of teaching any subject is the formulation of a definite statement of the purposes that are to be served by it. The teacher who has a clear conception of the ends which can be attained through the teaching of arithmetic has a definite goal towards which all energies must concentrate. He has, in addition, a standard of measuring the worth of his methods of teaching, for the effectiveness of a method of presentation is in direct proportion to its ability to achieve the end sought. With such a formulation of "aims" every endeavor of the teacher is governed by intelligent purpose and every end is attained with economy of effort. Teaching with a preconceived purpose leads to mental development; without it, to mental wandering.

The Values of the Study of Arithmetic.—A subject like arithmetic which has so long held a commanding place in the curriculum has developed a series of values to justify itself. These values can be grouped into two classes and summed up under five headings, viz.:

A. The Practical Values.
B. The Traditional Values, which include
   1. The Disciplinary Values.
   2. The Pleasure Values.
   3. The Cultural Value.
   4. The Preparatory Value.

An analysis of each of these values now presents itself for study.

The Practical Values of Arithmetic.—The utilitarian value of arithmetic has long served to explain its traditional place in the elementary school curriculum. The mere routine activities of daily life demand a mastery of certain number facts and number processes. In addition, a carefully selected curriculum in arithmetic helps to give children an insight into social activities that would otherwise be unknown to them. It is necessary that children understand the meaning of averages and the modes of computing them; that they know the meaning of common mathematical symbols, of percentages of increases and decreases, of rates of taxation, of interest, of insurance—in a word, the common business practices of social life.

The study of arithmetic must lead to a comprehension of life's problems from a quantitative viewpoint. We study current problems from an economic, a social, an artistic, or an ethical aspect. So, too, we must view these questions from a statistical or a quantitative angle. To know that the United States produces cotton, grain and lumber is to possess colorless geographical information; to realize that the United States produces four-fifths of the cotton, one-third of the grain, and one-third of the lumber of the world adds meaning and vital significance to these facts. To find that the attendance in a given school increased in one year from 2005 to 2995 gives no true appreciation of its growth, but to realize that the increase was almost one-half or about 50 per cent of the original attendance stirs the mind to a vivid

understanding of such phenomenal development. The fact that the earth revolves 25,000 miles in twenty-four hours does not give as vivid a realization of this remarkable speed as does the simpler statement that the earth revolves at the rate of a little more than one thousand miles an hour. Few people appreciate the significance of the statement that "the average wage in this city has increased from $450 to $500, while the prices of the necessities of life have increased about 30 per cent." Not until the wage increase has been reduced to a percentage basis, and the statement reads, "While the average rate of wages has increased 11⅑ per cent, the average increase in cost of food is 30 per cent," does the mind grasp the full significance of the fact. The utilitarian value of arithmetic reaches beyond the mere needs of elementary computation in the activities of everyday life. It adds a new viewpoint to experience, a new interpretation of life's problems and forces; and to that extent it enriches our understanding of them.

The Disciplinary Values of Arithmetic.—However vital we may regard the study of arithmetic as a preparation for everyday life, we have yet to justify the teaching of decimals beyond three places, areas of triangles and of parallelograms, areas of circles, and surfaces of prisms, pyramids and cones, indirect cases in percentage, the solution of imaginary quantitative problems—in a word, the long list of non-social topics that exist in modern school curricula. The inclusion of these socially unnecessary data has been justified since the development of modern psychological thought, on a purely disciplinary basis.

Statement of the Disciplinary Values of Arithmetic.—Tradition has reserved for arithmetic a central place in the course of study and has made proficiency in it a standard for determining general ability and promotion to higher grades. It is this tradition which has ascribed to arithmetic grossly exaggerated disciplinary possibilities.

The disciplinary values of any subject or experience are measured by the permanent effects that it can produce on the mind and character. Arithmetic has been given a high disciplinary coefficient because of the implicit faith that it can develop the powers of judgment, reason, concentration and abstraction as can no other school subject. It is obvious that most mathematical solutions necessitate deep thought and concentration to perceive the relationships that exist among various quantities. This unquestioning faith in the disciplinary powers of arithmetic holds that these mental powers, once developed, are agents for effective mental reactions in all situations where judgment, reasoning, abstraction or concentration may be necessary. Standard texts published during the last ten years voice with irritating assurance these possibilities, for they tell us:

Are you to be a lawyer? How can you learn to analyze a complicated legal case, if you cannot learn to analyze a simple proposition of geometry? Are you a student of history? How can you determine the influence of Napoleon on the world's development if you are incapable of determining the influence of a coefficient in a simple relation of algebra? Are you a linguist? How will you translate a masterpiece with its myriad shades of meaning, from one language into another, if you cannot learn to translate a trifling "reading problem" into the corresponding mathematical symbols? Are you to be a physician? How will diagnose and eliminate a disease, with its complicated, ambiguous and obscure symptoms, if you lack the faculties needed to diagnose and eliminate the unknown quantity out of an elementary equation?[1]

Mathematics typifies a method of reasoning . . . all life problems must undergo analysis before a solution can be obtained. . . . To be capable of handling a situation is power, regardless of occupation or social position. The study of mathematics is one of the most effective means to such an end.[2]

[1] Young, J. W. A., "The Teaching of Mathematics," page 34.
[2] Manchester, R. E., "The Teaching of Mathematics," page 19.

To these students of the psychology and pedagogy of mathematics, the mind is a vast reservoir of power, a storage battery of strength. Through the study of mathematics the mind is charged with this mental energy which remains there until it is to be utilized in any situation that may arise in life.

The Transfer of Mental Power.—Modern psychology and experimental education have long refuted this smug conceit. They teach us that mental power developed in one subject is not transferred to all other experiences, but only to those situations that are similar to the subject in which mental power was developed. The thought developed in arithmetic will help the child in the thinking situations in algebra or in the mathematical problems of physics, but will not aid him in the solution of problems in history, in politics or in home life. Arithmetic has no exclusive power to develop thought, abstraction and concentration. Properly taught, grammar, geography, nature study or domestic science can add to the child's power of thought and concentration; but here, too, there is no general transfer of power from one experience to other totally different experiences.

Experimental Evidence on Transfer of Ability in Arithmetic.—While the extreme doctrine of formal discipline has long been overthrown, the view was still held that ability developed in one branch of arithmetic can be utilized in all branches of the same subject. Recent experimental studies not only offer no evidence for such belief, but even tend to opposite conclusions. They show that ability in arithmetic is a sum total of abilities, a complex of powers; that proficiency in arithmetic must be resolved into proficiency in combinations, in copying accurately, in controlling many subprocesses like carrying, borrowing, placing, etc.; that most children are not proficient in all branches of arithmetic, but in one or more of these component branches. One

investigator,[1] who set before himself the problem of ascertaining whether numerical accuracy, i. e., accuracy in mere number manipulation, is transferred to accuracy in the reasoning problems of arithmetic, found that improvement in accuracy in pure computations seems to have produced no increase in accuracy in solving reasoning problems in arithmetic; that numerical accuracy should be made an end in arithmetic, not because it aids in the solution of reasoning situations in arithmetic, but because it is necessary in commercial and industrial life.

Experiments were made with two groups of children. Those in Group A were practiced in rule work and in mechanical number work. Those in Group B were without this special drill. In all other respects, the two groups were taught under identical conditions. Both groups showed slight improvement in reasoning work, but the rate of this progress was practically the same for both groups. To conclude that no transfer of accuracy takes place would be going beyond the limits warranted by the facts, for the accuracy developed in pure numerical operations aids in the numerical calculations involved in the solution of any problem. An investigator,[2] studying this problem of transfer of ability developed in one branch of arithmetic to another, concludes that such proficiency as seems to be transferred is due to a certain general ability which is developed, and which is capable of being used as much in the mechanical operations as in the reasoning problems. Heck [3] found that 50 per cent of the students standing highest in their law studies were

[1] Winch, W. H., "Accuracy in School Children: Does Improvement in Numerical Accuracy Transfer?" *Journal of Educational Psychology*, I. (1910), pages 557–589; II. (1911), pages 262–271.

[2] Starch, D., "Transfer of Training in Arithmetical Operations." *Journal of Educational Psychology*, II, pages 306–310.

[3] Heck, W. H., "Mental Discipline and Educational Values," page 97.

conspicuous for their poor work in mathematics, while 42 per cent of those who ranked among the poorest in the law school stood highest in mathematics. This cursory review of the data gathered by workers in experimental psychology and educational statistics offers enough evidence to cast serious doubt on the extravagant disciplinary hopes that were entertained for mathematics. We are told repeatedly that the study of mathematics inculcates the habit of neatness and system in work, develops a feeling for absolute accuracy, and cultivates a " reverence for truth, irrespective of sect, prejudice and self-interest." But the writer who makes these promises for arithmetic fails to add the necessary qualifying expression, " in all matters dealing with arithmetical solutions." The student who does his arithmetical work with system and neatness may not carry his neatness and system beyond the paper that bears the solution of the problems. The accurate accountant may indulge in falsehood, and the expert bank clerk may utilize his proficiency with numbers to conceal his personal dishonesty. To Young, mathematics can inculcate an esthetic sense, for " it has a beauty of its own, a symmetry and proportion of results, a lack of superfluity, an exact adaptation of means to an end which can be found only in the works of the greatest beauty." As an expression of loyalty to one's specialty and of faith in it, this statement commends itself at once; as an expression of actual fact, we must regard it as a superfluity of sentiment entirely at odds with the results of calm and dispassionate studies on the controlling values of arithmetic.

Summary: What Mental Power Does Arithmetic Give the Child?—The reader who has had a complete course in mathematics in his secondary and collegiate courses feels that the days devoted to this study gave him something definite and distinctive even though it cannot be formulated and labeled with scientific terminology. There can be no doubt

that mathematics makes its disciplinary contribution to the student's training. It gives him greater power to manipulate numbers and quantitative relations, capacity for thought and abstraction in mathematical problems, greater power of concentration, and ability to analyze a mathematical situation by studying the known conditions and discerning a relation among them which will lead to the value of the unknown element

Conclusion: Significance of Problem of Transfer of Abilities in Arithmetic.—The correct understanding of the disciplinary value of arithmetic is vital and far-reaching in its consequence. If power developed in mathematics is transferred to all other experiences, then we need not concern ourselves with the task of including in the course of study in elementary school arithmetic only such facts, processes, and computations as are used in actual social life. Any problem, however artificial, any mode of solution, however archaic, will serve to inculcate certain mental powers. But a clear comprehension of the problem and the extent of transfer of abilities leads at once to a realization that arithmetic must be more than mental gymnastics, that it must draw its content from commercial and industrial life, that apart from explaining social practice and answering social needs, it has no place or purpose in the course of study. Modern psychological studies on transfer of abilities have rendered no little aid in ridding the elementary school course in arithmetic of its scientific character, and in eliminating from it much of the unnecessary work in tables of uncommon weights and measures, complex fractions, calculations with fractions whose denominators are absurdly large, impossible reductions ascending and descending in denominate numbers, extreme rationalizations in early stages of number work—in a word, the socially useless matter that was retained because of false psychology and reverence for tradition. All these must go. The course of study must be simplified

by rigorously excluding all those topics that are not possessed of social use, and by so teaching the subject that habits of speedy and accurate manipulations and solutions are inculcated, while increased powers of thought, abstraction, concentration and analysis are natural mental by-products.

Pleasure Value of Arithmetic.—The modern curriculum has been liberalized to include more than the practical and the disciplinary subjects. Music, nature study, art and literature are taught primarily for their ability to give pleasure, and to enrich the life of the youth by teaching him how to invest his leisure. Many pupils experience a keen sense of pleasure in solving mathematical problems, in discovering the value of the unknown quantity. Few pupils intently at work on an arithmetical situation obtain the correct answer without feeling the thrill of joy that accompanies successful endeavor. The high school student utters his Q. E. D. with a feeling of deep satisfaction.

Instinctive Demand for Number.—Mathematical computations and solutions possess a pleasure quality because they satisfy certain instinctive cravings of the human mind. The economy of nature demands an instinctive mastery of the world on the quantitative side. The psychologist tells us that the barbarian aiming his crude arrow, unconsciously pulls the bow to the degree of tautness necessary to send the arrow a given distance. Should the prey suddenly start, the hunter unconsciously slackens or tightens the string on the bow, depending on the direction in which the animal is going. So, too, in every activity of the day's routine, in reaching out for an object, in pulling, tearing, breaking or lifting, man unconsciously strikes a balance between the energy necessary for the successful accomplishment of a necessary end, and the effort to be overcome. Because human wants are great in number and rich in variety while human energy is limited, the economy of nature has made the unconscious balancing of energy to be expended

and the resistance to be overcome, an inherent function of the mind.

Early arithmetical work is rich in material that satisfies the child's rhythmic sense. Children delight in number games and enjoy the rhythm of the count as in 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, etc., or the 5, 10, 15, 20, etc. Even adults find it hard to walk along beside a fence without touching every alternate picket, or to walk on a flagged sidewalk without stepping on every third crack. Teachers would do well to seek ways of utilizing the rhythmic sense in all work in counting, measuring, and tables.

Arithmetic also satisfies the puzzle instinct, or the instinct of curiosity. The child finds delight in pitting his skill against a puzzle, provided he feels that the solution is within his capabilities. Older people spend valuable hours on interesting puzzles that have no practical application. People who have reached years of discretion and whose thoughts usually go to the serious affairs of life invest their energies in attempts to find the cost of three herrrings if a herring and a half costs a cent and a half, to find the age of Ann, or to find how many bounds the hound must make to overtake the hare. Drills in multiples, factoring, finding the least common denominators, or greatest common divisors can be made interesting to children by presenting them in the light of puzzles. Once the instinct is aroused, pupils give dynamic interest and sustained attention even to purely mechanical processes in arithmetic.

To some children, numerical processes and symbols serve to stir vivid and quaint imagery. Pupils tell us that 4 represents a fat man and 7 a tall lady; that 9 calls up the picture of a lazy boy; that 6 tries to " break into 24 but 4 stops it and it finally breaks into 42 because 7 that was supposed to watch it was too weak to stop it." Most people express a definite preference for odd numbers in units place and even numbers in tens place. The numbers six

and seven and nine play important roles in mythology and early religious ritualism. Interesting studies made by experimental psychologists and varied evidence gathered by anthropologists serve to reinforce the belief that number has the power to stir imagination.

The unconscious desire to acquire a mastery of the quantitative phases of life, the rhythmic sense, the attractiveness of the puzzle, the image-making power of numbers—all coöperate to give the child a pleasure in mastering number facts and number processes. Teachers must aim to utilize this instinctive basis of number and to endeavor to relate their teaching of arithmetic to the child's fund of native interests. To neglect these is to reduce number study to a mechanical and arbitrary series of symbols and processes devoid of life and interest.

Cultural Value of Arithmetic.—Mathematics, like most subjects in the curriculum, must be taught because society demands the possession of a set stock of knowledge before it will approve the cultural standing of any individual. It insists that he know the meaning of certain terms and that he understand certain allusions. In reading the daily newspapers one finds expressions like, " as basic as the square on the hypothenuse," " the larger the divisor, the smaller the quotient," "a changing ratio," "a proportional relation," " the means and the extremes." But care must be taken not to allow the demands of traditional culture to overshadow the practical value of arithmetic. While the cultural value must be taken into consideration, it must, at no time, assume a controlling force in the formulation of a course of study.

Preparatory Value of Arithmetic.—A final value which is often urged for arithmetic is its ability to prepare the child for later studies. The child who knows the fundamentals of arithmetic can solve the problems that arise in physics and can grasp with greater ease the basic operations of algebra. While this preparatory value of

arithmetic must go unchallenged, it must always be borne in mind that it applies only to a limited number of children. The vast majority of our elementary school pupils are not destined to enjoy the advantages of a high school education. To them the preparatory value of arithmetic is non-existent, and it must play no part in shaping their course of study.

Summary and Conclusion.—Our study has sought to justify the teaching of arithmetic on a fivefold basis: the practical, the disciplinary, the pleasure, the conventional and the preparatory values. But we must be sure not to see in these justifications of arithmetic five coördinate values. There is a relativity of worth among these values that must be seen clearly and kept in view as the guiding principle in teaching and selecting the subject matter of arithmetic. We are rapidly making for a new arithmetic. The new psychology which opposes the doctrine of the transfer of abilities, the view of education as a socializing function, the demands of industry, the more sympathetic comprehension of child life—all these are coöperating to humanize the subject and to teach that the practical value of arithmetic is the primary value. To it, all other values must bend. A course of study in arithmetic, selected and organized with the utilitarian aim in view, can be so taught that all other values are attained in their fullest measure.

CHAPTER II

THE COURSE OF STUDY IN ARITHMETIC

The Course of Study a Determining Factor in Efficient Teaching.—It must seem obvious, even to the lay mind, that the formulation of a course of study in arithmetic is the basic factor in determining the efficiency of the school course in this subject. If the course of study errs in the selection of subject matter, teachers and pupils must spend valuable time on what has no social worth; if the course of study is organized with no regard to sequence and gradation, the children's minds are confused by the rapid succession of difficulties; if the course of study does not set a proper proportional allotment of time to mechanical and thought exercises in arithmetic, children are over-drilled in the fundamental operations and are rendered helpless in arithmetical situations that deviate from the type form; if the course of study sets a fixed limit for all children in a given grade, some are discouraged by being required to attain a standard needlessly high, while others, more gifted, are meanwhile marking time. In the final analysis, the course of study determines the possibilities of introducing effective modes of instruction and of developing such arithmetical ability in our school graduates as will meet the approval of the business men who employ them.

The Course of Study Must be Definite in Its Requirements.—Much confusion arises from general statements and indefinite requirements in a course of study in arithmetic. It is not sufficient to prescribe for a grade the following work:

Oral.—Daily drills to develop power, accuracy and alertness. Oral work to review, introduce and supplement written work, involving both abstract numbers and concrete problems. Terms—base, rate and percentage.

Written.—Integers—Daily practice in the fundamental operations, with attention to accuracy and rapidity.

Fractions.—Addition, subtraction, multiplication and division of fractions and mixed numbers. Complex fractions. Principles: multiplying numerator, multiplies fraction; multiplying denominator, divides fraction; dividing numerator, divides fraction; dividing denominator, multiplies fraction.

The course of study must be more definite and more concrete in its prescriptions. Each succeeding grade makes the same demands for " power, accuracy and alertness," yet no teacher of any grade knows to what degree of " power, accuracy and alertness " she is expected to bring her pupils. The course of study at no place defines its meaning of these three terms. It is obvious that all teachers must have the same interpretation of these three aims and must be told more definitely what degree of each of these qualities is to be attained in any grade. Teachers must be told that in a specific grade it is expected that the children will develop the ability to add 12 lines of figures, each of 3 places in x minutes; that of any ten such exercises, a pupil is expected to attain y per cent of accuracy; that in fractions children are expected to be able to work examples like the following: $\frac{1}{8}+\frac{1}{4}$, $2\frac{1}{3}+\frac{1}{6}$ (i.e., in which common denominators can be found by inspection), and to solve practical fractional problems based on the types submitted. " Fractions," as a topic, is too inclusive to carry definiteness of meaning; " complex fractions," when assigned without explanation, will lead some teachers to teach nothing more complex than $\frac{3}{5}/\frac{1}{5}$, while others, carried away by the disciplinary value of arithmetic, will require children to master the solution of examples like $\frac{\frac{2}{3}\times\frac{5}{6}\text{ of }\frac{1}{4}}{\frac{3}{4}\times\frac{7}{8}\text{ of }1\frac{2}{3}+\frac{5}{9}}$. Both are

complex fractions, and neither is vitally necessary for the average elementary school graduate. These indefinite assignments cause time to be wasted and energy to be dissipated. They also discourage teachers and pupils because too high a standard may be set by the teacher of any grade.

How Can Definiteness of Course of Study be Produced?—Let us now consider a few simple devices which give a course of study a degree of definiteness that will help to insure concentrated effort towards clearly conceived ends.

1. Terms must be defined so that all active agents in the same school system have the same conception of the exact requirements of the course of study. Courses of study ask teachers to develop "alertness," "power," "good arithmetical habits," "proper algorisms," "modes of thought in arithmetic," "modes of attack," etc., but fail to provide for the fact that these terms and expressions have no fixed connotation and vary with each teacher according to past experience and training. "Alertness" may mean speedy mental reactions to some, and mental sagacity to others.

2. Quantitative norms and concrete illustrations must be given as the approximate minimum limits to be expected in successive grades. Assuming that the terms taken from courses of study are actually defined and illustrated, it is evident that the degrees of proficiency vary with each succeeding grade. Accuracy in addition in the fourth year is far below that expected of a sixth year class but not so far below that expected of a fifth year class. How is a teacher to know what degree of neatness, of accuracy, of speed, of power to systematize, of ability to meet new arithmetical situations, is expected of her pupils at the end of a school term or is to be expected by her at the beginning of a grade? Specimens of children's papers ought to be submitted in the course of study so that teachers will know the degree of neatness that is expected each succeeding year. The course of study must say, "at the end of this term's work it is expected

that children will attain the power to solve x examples of the type of 152×16 in y minutes and attain an accuracy of z per cent." Experimental work in establishing quantitative norms in arithmetic has gone far enough to warrant progressive school authorities in setting up the best of these as tentative standards for their school system.[1]

It would be profitable to compute, if it were at all possible, the amount of energy that would be conserved by such definite standards. Teachers are working in obedience to general directions without means of measuring their rate of progress and without knowing exactly when they have attained the normal proficiency for children of a particular age. To push ahead of certain normal attainments that can be achieved by normal children working under normal conditions may cause a strain resulting in irreparable harm. S. A. Courtis in his "Practice Tests" gives a table of quantitative standards showing the degree of proficiency in addition that can be developed by children in each school grade. Mr. Courtis has devised a graded series of lessons in addition, passing from the very simplest case to very difficult examples. He has determined by testing thousands of children under experimental conditions that fourth year children can be so taught that they are able to add 72 single column examples, each of 3 numbers, e.g.,

$$\begin{array}{ccc} 6 & 8 & 4 \\ 3 & 4 & 7 \\ 1 & 7 & 4 \\ \hline \end{array}$$

in 6 minutes; that sixth grade children can do the same in 4 minutes, and seventh grade pupils in $3\frac{1}{2}$ minutes. These are tentative norms, applicable to the Courtis Practice Tests, and suggest a mode of developing similar ones in the other operations of arithmetic.

[1] For detailed study of quantitative norms and scientific evaluations, see Chapter XVII.

3. Throughout the assignments, types of problems, solutions, and arrangements should be suggested. To assign "Percentage—Case III—Problems in percentage, to find the base, given rate and percentage," will lead only to general confusion. Some textbooks group all cases in percentage under three headings, others under five. According to some standard texts the example, "Goods sold for $120 brought a profit of 20 per cent; what was the cost?" is the type referred to in the above assignment; other books, equally authoritative, would group this under Case IV and allow only an example like the following under Case III—"Goods sold at a profit of $36 brought me 12 per cent gain; find the cost." It is obvious, therefore, that the course of study must add a number of typical examples as concrete illustrations of its assignments.

4. Definiteness of assignment can be attained by constant references to standard textbooks that are in the school libraries or on teachers' desks. Typical extracts from some of the better courses of study used in various cities read:

Reduction of common fractions to decimals. Use no fraction with numbers larger than those in Y. and P.'s Arithmetic, page 54.

Find L. C. D. Do not be more intensive than is required to solve examples in W.'s Arithmetic, pages 74–76.

Addition of Fractions. Use form suggested in S.'s Arithmetic, page 93.

5. When a subject is to be taught through a number of grades, there must be a limited assignment for each grade. One finds, in courses of study, topics like, "Business Forms," "Tables of Weights and Measures," "Abbreviations commonly used in Business," etc., in the assignment for a number of grades. If business forms are to be taught in 5A, 5B and 6A, it is obvious that each grade teacher must know the limits of her responsibility. Under the general assignment, each teacher teaches all the business forms so that the pupils

may be ready for any business form that may be given in the examination set by her superiors. With this undue increase in the scope of work in any grade, time for proper drills and for habituation is insufficient and the necessary facts and forms are imperfectly taught, only to be retaught in the succeeding grades where the strain of the situation continues.

MINIMUM AND MAXIMUM COURSES

Flexibility of Course of Study.—A sound course of study in arithmetic allows the teacher of any grade an opportunity to adjust the subject matter to the needs and the capabilities of her pupils. It is not necessary nor possible to teach all pupils the same number of forms and processes in fractions or decimals. Children who, in the normal course of events, leave school at the end of the sixth grade may be spared the knowledge of complex fractions or of decimals beyond three places. All indirect cases in problems, in fractions, and in decimals can well be omitted from this course. So, too, the child who is woefully behind in arithmetic need not be given as intensive a study of any topic as the child who shows power and ability in dealing with quantitative relations. In addition to assigning a definite number of topics and processes for a given grade, the course of study must indicate the irreducible minimum of facts, of habits, and of solutions in which all children are expected to attain a high degree of proficiency. Children of arithmetical ability should not be held to this minimum but should be taught the course of study with a degree of intensiveness proportional to their capacity. A cause of waste of energy and of discouragement to both teacher and pupils is found in the habit of trying to bring the entire class, regardless of ability or actual need, to the same level of proficiency in all phases of a subject. The course of study must not only allow but must encourage discrimination and differentiation of

subject matter so that it attains a high degree of adjustability.

A concrete case may make this suggestion clearer. The course of study in arithmetic in one of our large cities prescribes for the 5B grade (fifth year, second half) the following topics:

COURSE OF STUDY

GRADE 5B

Special Work: Decimals

Oral

Daily drills to develop power, accuracy and alertness. Oral work to review, introduce and supplement written work, involving both abstract numbers and concrete problems.

Terms—**Common fraction, decimal fraction, concrete number, abstract number.**

Written

Notation and numeration—Three periods of integers; **five orders of decimals**; relation of the orders of integers and **decimals**; **effect of moving the decimal point.**

Integers—Daily practice in the fundamental operations, with attention to accuracy and rapidity. Principle: **multiplying or dividing both dividend and divisor by the same number does not alter the value of the quotient.**

Common fractions—Additions, subtraction, multiplication, and **division. Fractions commonly used in business. Mixed numbers.**

Decimals—Reduction of common fractions to decimals, and of decimals to common fractions. Decimal equivalents of halves, fourths, eighths. Addition, subtraction; multiplication of integer by decimal or decimal by integer; division of decimal by integer.

Problems—Finding one or more of the equal parts of a number; finding what fraction one number is of another. Application of these types to concrete exercises. **Calculating by the hundred, thousand, dozen, and fractional parts of a dollar.** Use of denominate units.

Business forms—Bills for **several articles** purchased **at different times**; receipting bills.

SYLLABUS

GRADE 5B

When the divisor contains a decimal, it should be changed to a whole number by removing the decimal point and making a corresponding change in the dividend. The decimal point in the quotient should then be written over the decimal point of the dividend as changed.

An analysis of the arithmetical requirements of the child entering the business world shows that decimals of three places will meet the necessities of ordinary situations. Decimals of three places should therefore be made the minimum requirement. For strictly utilitarian purposes it is not necessary to change fractions like $\frac{7}{27}$ or $\frac{1}{14}\frac{3}{3}$ to decimals; all the decimal equivalents necessary for the day's computations can be memorized after solving them with the aid of the monetary system. Not all children should be required to learn how to change any common fraction to a decimal, but all children should know the decimal equivalents of $\frac{1}{8}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, etc., fractions used in business and possessing denominators that bear a simple ratio to a decimal denominator like 10, 100 or 1000. So, too, it may be well to teach what part one number is of another, but there are so many variations of the type that it becomes necessary to differentiate examples, under this case, that must be taught to all children, from those that may be given only to a selected group. All normal children in the 5B grade can be taught to find "what part 8 is of 16," "what part 5 is of 9," or, "if I had 240 lbs. and sold 80, what part of my stock did I sell?" But not all children can solve problems of this case that vary the conditions, viz., "I sold 90 lbs. and had 160 lbs. left; what part of my stock did I sell to-day?" Mathematically speaking, this last problem belongs to the same case as the preceding one, but it is obvious that only the child of arithmetical capability should be required to study this type so intensively. The course of study must indicate in every grade the minimum requirement, so that children whose school education is unduly short or who have little ability in quantitative computations may be given that in arithmetic which will prepare them for the demands of society.

Cautions in Formulating Minimum Course of Study.— The advantages of setting up a minimum expectation for each grade may often be conterbalanced by serious consequences.

There is always great danger that the minimum becomes the maximum requirement and the class as a whole is again reduced to the level of the mediocre or the backward. Great care must be taken in the supervision of a school to make certain that the course of study is adapted to the needs and abilities of the pupils. No child should be limited to the minimum course of study without the consent of the principal or his assistant or without notifying the parents.

A second great danger in formulating a minimum course of study in arithmetic is that the maximum course may be made up of those topics and processes that have no social use and that are taught for traditional, disciplinary or cultural reasons. We must ever be mindful of the fact that in the last analysis, the same standard—social need—determines the content of both the maximum and the minimum course; the difference is merely one of degree. If the child who leaves school at the end of the sixth year cannot be taught all that is socially useful in arithmetic, he can be taught the most useful processes and facts. If the child's mental equipment will not enable him to assimilate, within the time limits of the school course, all that is commercially and industrially necessary in arithmetic, he can be taught what will be of relatively greatest need in his future life. At no time must the maximum course be permitted to degenerate to formal mental gymnastics, puzzle solving, and memorizing obsolete tables and processes. The child studying the maximum course must be given: (1) a deeper insight into numerical relations, (2) a wider knowledge of socially necessary arithmetic, (3) a high degree of skill, increased speed, and greater accuracy in the fundamental solutions.

THE CONTENT OF THE COURSE OF STUDY

We come now to a more specific and more detailed analysis of the content of the general course of study and of the specialized maximum and minimum courses. We shall con-

sider: 1. The Minimum Course of Study; 2. The General Course of Study; 3. What to Omit from the Traditional Course of Study; 4. Algebra and Geometry in the Course of Study; 5. Industrialized and Commercialized Courses in Arithmetic.

1. The Minimum Course of Study.—Selecting on the basis of relative social need, we find an irreducible minimum without which a child cannot understand the most common quantitative aspects of his environment nor intelligently approach the arithmetical tasks that make up the day's routine in the usual activities of life. The absolute minimum can be summed up in the following table:

A. Fundamental operations (addition, subtraction, multiplication, division) with whole numbers.

B. Fundamental operations with common fractions. The denominators same as those in usual business fractions.

C. Fundamental operations with decimals. Limited to three places.

D. Problems:

Fractions; to find fractional part of whole number, fraction, mixed number.

Decimals; to find decimal part of whole number, decimal, mixed number.

Percentage; to find per cent of any reasonable quantity.

Finding cost of a quantity, given the rate per 3, 5, dozen, 100, 1000.

Simple Proportion.

E. Percentage:

Direct case (First Case) of any of the common business applications of percentage, viz., profit and loss, commission and brokerage, interest (years and months), commercial discount.

F. Business Forms:

Bills, receipts, receipted bills, checks, stubs in check book, deposit slip, cash book account.

G. Tables of Weights and Measures:

About six of the commonly used tables. Selection of the

tables to be determined by needs of the locality. Simple computations with these quantities.

H. Mensuration:

Finding area of rectangular figures. Finding volume of cube, box, room.

I. Special application of fundamental processes to industries and business of the immediate environment.

This tabulation gives merely a bold outline of what may be considered only a tentative basis of a minimum course of study. It will be necessary to allot the various parts to the different grades; to correlate some of them in any one grade so that the class learning to multiply whole numbers and fractions will also learn how to apply this knowledge in making out bills; to define with great clearness how intensively each of these topics is to be taught; to set up a concrete or a quantitative measure of the approximate degree of proficiency to be attained at the end of any one year. This outline, it must be reëmphasized, is not a course of study; it is only a guide in formulating the minimum arithmetical requirements.

2. The General Course of Study.—The supervisor engaged in formulating a general course of study may find it helpful to have before him a summarized or a composite course of study containing the constituents of the latest courses in large cities or in leading states. The following summation indicates the content of representative courses, year by year:

First Year: Incidental number work. Taught through correlation with games, music, manual training.

Degrees of variations: From no number work to formal drills in addition, subtraction, multiplication.

Second Year: Number work through correlation. Addition and subtraction emphasized. Multiplication tables begun.

Variations: From no number work to drill on simple division.

Third Year: Addition and subtraction mastered. Much time to multiplication and division tables.

Variations: From mere introduction of arithmetic to considerable drill in fractions that can be added and subtracted by inspection.

Fourth Year: Multiplication and division mastered. Fractions by inspection.

Variations: From mere introduction of formal computations and incidental work in fractions to formal work in fractions.

Fifth Year: Fractions mastered. Decimals introduced; denominate numbers.

Variations: Too wide to be formulated.

Sixth Year: Decimals mastered. Percentage and profit and loss. Problem work. Denominate numbers.

Variations: So great as to permit of no systematization.

Aim of the First Six Years: To develop skill in the fundamental operations as applied to whole numbers, common and decimal fractions, and to teach the necessary number facts.

Seventh and Eighth Years: Percentage and every variety of business applications. Mensuration applied with great variations in extensiveness of treatment, greatly reduced in best courses of study. Ratio and proportion. Longitude and time. Metric system. Square and cube root (not in progressive courses of study). Second half of the eighth year often devoted to a complete review or to algebra and geometry.

Aim of Last Two Years: To develop power to understand arithmetical solutions and manipulate quantities in problems varying from the type.

Summary.—This composite course of study, the constituent courses of which represent the best efforts of supervisors in representative communities of the United States, shows us clearly that the formulation of courses in arithmetic still proceeds on the basis of personal opinion and traditional demands. There is little indication that those charged with the making of the course first formulate the aims of the course in arithmetic as an intelligent basis for selection of material. Nor is the sequence of subjects or the gradation controlled by any greater scientific considerations. The worker in experimental education and psy-

chology has here a virgin field for investigation. The chaos of aims and organization that characterizes the work of the last two years has contributed no little to the discontent with the present system of an eight-year elementary course and its reorganization into the Six-Six plan.

3. What to Omit in the Traditional Course of Study.—An analysis of existing courses of study shows clearly that most school systems have not freed themselves from the tyranny of traditional demands in arithmetic. Progress is retarded by much that is archaic or useless in actual trade or business practice. What can be omitted from prevailing courses of study? The following table suggests an answer to this oft-repeated query:

A. In Fractions.

(1) All fractions whose denominators are larger than sixteenths, or, if smaller, are unusual in actual occurrence, e.g., fifteenths, thirteenths, sevenths, elevenths, etc.

(2) Greatest common divisor and least common multiple or denominator that cannot be found by inspection.

(3) Complex fractions.

B. Denominate numbers.

(1) All uncommon tables of weights and measures, e.g., Troy weight, Apothecaries' weight, Surveyors' measure, etc.

(2) Units of measure, obsolete, imaginary, or not used in the child's environment, e.g., long ton, long hundredweight, small measure (half bushel), hand, rood, furlong, etc.

(3) Reduction ascending, descending, division and multiplication of denominate numbers involving more than three units, e.g., "Reduce 1 mile, 2 rods, 3 yards, 2 feet, 7 inches to inches." "Reduce 5,246,489 inches to high denominations." "How much will 72 barrels hold, if the capacity of 1 barrel is 15 gallons, 3 quarts, 1 pint and 2 gills?"

C. Percentage.

Unusual and obsolete business applications; partial payments; true discount; interest in cases where the rate is not one used in actual business or in cases where principal must

be found; stocks and bonds in classes not specializing in commercial arithmetic; life and fire insurances computed by rate per cent on the dollar, thus: "Find premium on a fire or life insurance policy, the rate being $\frac{3}{5}$ per cent."

D. Problems.

(1) Indirect Cases. Great disagreement among teachers. Where indirect cases are taught, children not up to grade work should not be required to master more than the direct cases.

(2) Partnership. Corporate organizations more common, hence partnership no longer used. Mode of computation to be the same as that actually followed in business.

(3) Compound proportion: Examples of this type: "If 54 men working 20 days build a wall $750 \times 15 \times 2$ feet, how many men are required to build a wall measuring $250 \times 12 \times 2$ feet in 30 days?"

(4) Problems that form no part of industrial and commercial practice but are given as mathematical puzzles to challenge thought and ingenuity.

E. Mensuration.

Areas and volumes of uncommon figures, e.g., area of trapezoid, surface of cone, sphere, etc., volume of cone, pyramid, sphere, frustum of cone, etc. Such work should be part of the content of geometry or part of vocationalized arithmetic.

F. Vocationalized arithmetic.

The technical procedures and computations of specific and specialized industries cannot be made the subject for study by all children. Only those phases of arithmetic which contribute to the understanding of the common experiences of life should be incorporated in a general course. Examples in plastering, carpeting, paperhanging, gilding, etc., are subject to specialized trade practices and should be omitted.

G. Miscellaneous subjects.

(1) Square root of number too large to be solved by inspection or factoring.

(2) Cube root.

(3) Longitude and time beyond simple case of computing approximate difference, allowing 15 degrees to an hour.

Courses in arithmetic have grown gradually but continuously in content until the channel of subject matter has increased in width but has suffered a proportional diminution in depth. In formulating the table of what can be profitably deleted from prevailing courses of study, the author has suggested only those subjects and processes on which authorities are generally agreed. Too much has been attempted in arithmetic and the result has been an inevitable superficiality of understanding and lack of skill in manipulation in fundamental operations. Simplifying the course of study by eliminating that which is of comparatively little social use will give time to narrow the channel of knowledge so that thoroughness of comprehension and skill of manipulation can be attained.

4. Algebra and Geometry in the Elementary School Course of Study.—A. *Confusion as to the Place of Algebra.*—Much of the confusion in arithmetic in the upper grades can be traced to the introduction of algebra as a separate and distinct subject in elementary school mathematics. About 35 per cent of our city school systems teach algebra as an unrelated branch of mathematics. The topics taught in algebra are usually those found in high school courses of study. A simple textbook is used as a guide for the subject. Some courses complete the arithmetic by the end of the seventh year; others by the end of the sixth year. The last two or three or four grades of the eight school years are devoted to algebra and geometry with no attempt to establish any vital relation between these new branches of mathematics and the arithmetic which preceded them.

B. *Purposes of Adding Algebra to the Elementary School.*—A variety of reasons has been offered to justify the inclusion of algebra in the elementary course of mathematics. Briefly the justifications may be summed up under five heads, viz., (1) to aid pupils in analytical solutions by giving them mastery in the use of literal elements; (2) to aid pupils in

interpreting industrial formulæ; (3) to enrich the course in elementary school mathematics; (4) to interest pupils in the work of the high school by giving them glimpses of higher education; (5) to bridge the chasm between elementary school and high school curricula. To these, Smith adds, (6) to train pupils in the logic of mathematics and (7) to give pupils the ethical sense which mathematics can inculcate.

C. *The Proper Place of Algebra in an Elementary Course.* —Whatever merit these reasons may have, it is not sufficient to justify the teaching of algebra as a separate subject. Pupils should be given skill in using literal elements and in interpreting simple formulæ, but this is an end which can be attained by teaching only the simple equations of algebra and by making this part of algebra an integral part of arithmetic. The course in arithmetic can be more properly enriched through varied applications to commercial and industrial life, rather than through condensing the course and adding a new branch of mathematics. Whether teaching algebra in the elementary course will bridge " the dreaded chasm " or give a new interest in high school work is extremely doubtful The disciplinary and the ethical values of algebra can be derived from arithmetic.

All literal elements must be introduced as natural parts of the general course in arithmetic and must serve to make the solution of problems speedier and more accurate. The child finds no difficulty in using symbols early in reading and in writing. So, too, children readily acquire power to symbolize relations, operations, and quantities in arithmetic, for all analytical solutions of problems are, in the last analysis, literal or algebraic solutions. The child who solves the problem, " A third of my ruler was broken; I lost the use of 6 inches; how long was the ruler? " by explaining, " The ruler was $\frac{3}{3}$ long; $\frac{1}{3}$ of the ruler is 6 inches; $\frac{3}{3}$ of the ruler is 6 inches$\times 3$," is using an algebraic method.

Substitute x for "ruler" and the solution acquires the accepted algebraic form. Uniform literal symbols and uniform solutions of simple equations having been taught, algebra will have made its complete contribution to elementary arithmetic. In the chapter on Problems the details of the inclusion of algebra in arithmetic and the methods of teaching the various forms of the simple equations will be fully considered and illustrated.

D. *Geometry in the Elementary Course.*—Disapproval of the method of teaching mensuration in arithmetic led to the selection and simplification of a body of geometrical facts and their addition to the elementary school course. Children were too often taught to memorize definitions of geometric forms, formulæ for obtaining areas and volumes and other procedures of solution, through severely deductive lessons devoid of all concrete experience. The revolt against such unpedagogical practices led to a curtailment of this branch of arithmetic and the gradual introduction of inductive lesson-forms and the correlation with manual training and drawing lessons. This brought simplified geometry which was called "inventional" or "concrete" or "observational" geometry and which sought to develop the laws of areas and volumes through experience in constructing the common geometric forms. But the important principle to keep in mind is not to group all this geometric work into one grade, but to introduce it gradually and make it coextensive with the course in arithmetic. The great danger of assigning all this work to one grade in the upper part of the school course is that it then becomes, not a natural application of the fundamental processes of arithmetic, but a superadded, emasculated summary of the formal geometry taught in the first high school year. The content of this inventional geometry and the characteristic modes of teaching it must be reserved for a later chapter on Mensuration.

5. Course in Industrial and Commercial Arithmetic.—The youth of the elementary school is in greater need of industrialized and commercialized arithmetic than of a course in industrial and commercial arithmetic. The preceding paragraphs show clearly that the function of the school is to develop that ability and those habits in number which will enable the child to approach arithmetical situations intelligently. It is not the function of the general school course to give a detailed study of the arithmetical practices in any one or in any group of specific industries. That course in arithmetic which relates itself to *all* the activities of the child's life and especially to *all* the industries of his environment is as commercialized and industrialized as any school course need be.

THE DEVELOPMENT OF A COURSE OF STUDY IN ARITHMETIC

The study of the course in arithmetic has thus far concerned itself primarily with the selection of the content. Assuming that problem solved, the next question is how to organize and unfold the many topics in this subject. Pedagogical theory and experience have evolved three solutions:

1. Simultaneous Presentation of the Fundamentals.—Grube, one of the earliest students of the problem of organization, suggested the simultaneous presentation of fundamental processes in arithmetic. When the pupil learns to add $2+2+2+2$, he should be taught 2×4 and $8-2$ or $8-4$; knowing 2×4, the division of 8 by 2 or by 4 should also be introduced. It is argued that these processes mutually supplement one another and build up in the child's mind rich concepts of the numbers so taught. Whatever the theoretical advantages may be, it is obvious that such a system of organization applies only to small numbers, for with large numbers only confusion will result. The fundamental processes whether applied to whole numbers or to fractions are surely not of equal difficulty. Frequently, one presupposes

a mastery of another; in division by a number as small as 27, a knowledge of multiplication and subtraction is necessary. The practical teacher insists, therefore, on a sequence which shows at least a conscious attempt at rational gradation. When applied to branches of arithmetic other than the fundamental operations, this system of organization falls of its own weight.

2. Exclusive and Successive Attention to Each Branch of Arithmetic.—The obvious organization counsels attention to one process or one branch of arithmetic at a time. When that is mastered a second part of the subject is taken up. The prevailing course of study in arithmetic up to about 1895 or 1900 taught notation and numeration in one grade, addition in the next; similarly, subtraction, multiplication, division, fractions, denominate numbers, decimals and percentage were taken up, one by one, and completed, before the next was attempted. In the "addition grade" children added any quantities within reason. In the "fraction class" or classes, every possible manipulation with fractions was taught so that vigorous and persistent drill was assured.

The old schoolmasters justified this organization on many grounds. They argued that it gave opportunity for thorough grounding in essentials; that the organization provided for necessary drill; that concentration on a single line of thought was encouraged; that weaknesses in arithmetical knowledge of pupils could be traced forthwith to the proper grade, and thus responsibility was fixed; that the child realized clearly that he was acquiring a new branch of the subject; that there was a definiteness of assignment too frequently absent in present-day courses.

That there are advantages in an organization of the course which makes for exclusive attention to each part of the subject must be admitted by every opponent of the old system. But it nevertheless has inherent weaknesses that must not be overlooked. Exclusive attention to each of the

succeeding branches of arithmetic tends to mechanize the subject. The richness derived from a course which contains variety of application is lost. The various branches of arithmetic are not mathematically independent but are interrelated, and when properly correlated contribute towards clear comprehension. Under this system of artificially partitioning the subject, the child did not realize that $\frac{5}{6}$ and $.83\frac{1}{3}$ or $83\frac{1}{3}$ per cent are all the same quantity. Decimals were not fractions and percentage forms were not decimals. It is obvious that the subject as a whole must be taught as interrelated functions and that its component parts must not be isolated.

This system of exclusive attention to each topic in arithmetic is responsible for lack of intelligent gradation. In a general way, it is true that subtraction is more difficult for the child than addition and that multiplication ranks next in the scale of difficulty. But in the development of these processes, this succession of addition, subtraction, multiplication, division, etc., represents no gradation. It is much simpler for the child to learn to subtract 32 from 78 than it is to add ten numbers of five orders each; it is simpler to multiply 23 by 3 than to subtract 243 from 702. Similarly it is more intelligent gradation to teach the child to divide 936 by 3 than to postpone this type of example until he can multiply 7394 by 608. There is no reason to justify the teaching of all the cases and the forms of one process before taking up the next. A vivid realization of these limitations in the method of "Exclusive Attention and Successive Attention to Each Branch of Arithmetic" led to the modern development of the subject.

3. The Spiral Unfoldment.—To retain the elements of strength that are possessed by the two systems discussed above, the *spiral system* was evolved. It is a compromise method and seeks to give in successive grades a series of repetitions of the same branches of arithmetic but in constantly and gradually widening circles.

A detailed illustration of the spiral plan may give clear comprehension of this method of developing a course of study. The accompanying tables show the development of arithmetic through the grades in New York City schools. If the reader will take any topic of arithmetic, addition, notation and numeration, fractions, or decimals, and follow it through the grades down the vertical column, he will readily see that it is not developed completely in any one grade, but is repeated in more extensive circles as the course progresses. The reading of the total grade assignment by studying any one of the horizontal lines will show at once the variety of work in any one grade and the round of progressive repetitions.

The subjoined graphs, Figs. 1, 2 and 3, show us how many grades are required to teach the whole topic of decimals,

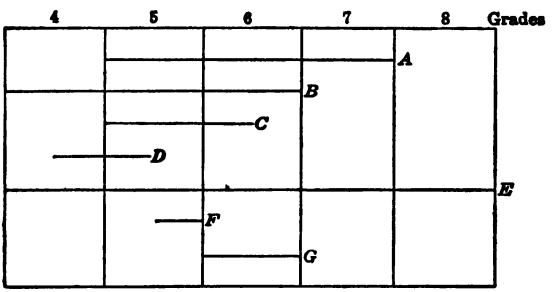


FIG. 1.—Graphic Representation of Grades in which Common Fractions are Taught.

Line *A* represents a school system which teaches fractions in three years, line *B*, in three years, but not the same years; line *C*, 1½ years; line *D*, in 1 year (from middle of the 4th year to middle of 5th); line *E*, in 5 years; line *F*, in one-half year; line *G*, in 1 year.

Courses represented by *A*, *B*, *E* are very spiral in development; courses represented by *D*, *F*, *G* are practically opposed to spiral forms.

percentage, mensuration or fractions. We see again that there is a complete absence of agreement among makers of courses of study and that local opinion seems to govern in

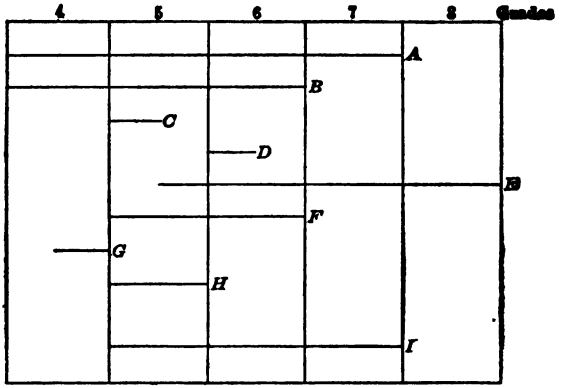


FIG. 2.—Graphic Representation of Grades in which Decimals are Taught.

Lines *A*—*I* represent different school systems. Courses represented by *A*, *E*, *I* are evidently very spiral; by *B* and *F*, moderately spiral; by *C*, *D*, *H*, opposed to spiral plan.

Note differences: Course *G* teaches decimals in second half of the 4th year; *C*, in first half of 5th year; *D*, in first half of 6th year. Course *A* requires 4 years; courses *C*, *D* and *G*, only one-half year.

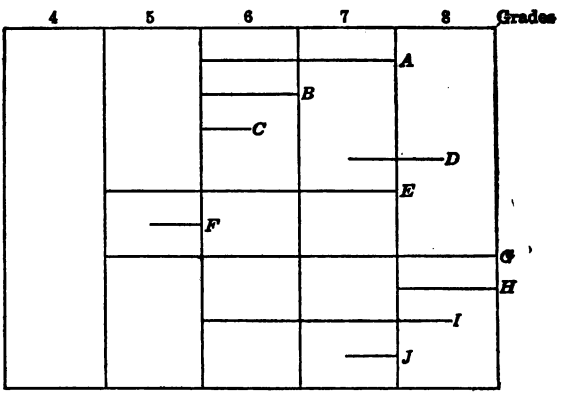


FIG. 3.—Graphic Representation of Grades in which Percentage is Taught.

Lines *A*–*J* represent school systems. Point out differences in time required by different systems. Which are extremely spiral? Moderately spiral? Opposed to Spiral Plan? Compare courses *F* and *J*, courses *F* and *G*.

allotting the various branches of arithmetic to the different grades.

Advantages of a Spiral Unfoldment.—The advocates of spiral courses in arithmetic set forth many advantages for them: (1) They argue that pupils who leave school before the completion of the eighth year are introduced to a richer and a wider range of topics in arithmetic. A cursory inspection of the tabular arrangement of the work of the grades in the New York City schools brings conviction on this value of the spiral system. (2) Greater correlation of the various branches of arithmetic is made possible, for in one grade the child learns that common fractions, decimals, and percentage are mathematically the same despite their difference in form. Children learning to add and to multiply fractional quantities apply these processes at once to bills, mensuration, denominate numbers and various problems. Such correlation makes for natural and varied drills and applications. (3) But this correlation leads to more rational associations, and therefore better comprehension and more lasting impressions result. (4) All spiral courses are replete with opportunities for reviews of the work of preceding grades. The child who could not master the initial concepts of decimals in the 5B grade is not permanently deficient in this branch of arithmetic, for decimals will be taught again in the 6A, and the 5B work will be reviewed as an apperceptive basis for the lessons in the higher grades. (5) Another important advantage in a spiral system is found in the fact that the subject is better graded, is more psychological than logical, and therefore insures better mastery in the long run. As the child grows older and gains in mental power, the various branches of arithmetic gradually increase in difficulty. In the older system, when all of fractions was taught in one grade or in one year, the child was early forced to learn the difficult forms and processes in fractional computations. But many children were not mentally matured for such thor-

TABULATION OF PART OF NEW YORK CITY COURSE

GRADES.	NOTATION AND NUMERATION.	ADDITION.	SUBTRACTION.	MULTIPLICATION.	DIVISION.
1A	Reading 1 to 100 Counting: to 10 with obj. 1 to 20 } no 10 to 100 } objects	Within 1 to 9 2 to 8			
1B	Reading: 1 to 100 Counting: to 20 with objects by 1's to 50 } No by 10's to 100 } obj. by 2's to 20 }	Add 10, 2, and 3 to any No. from 1 to 10. Sign +	By addition process. Minuend up to 20. Limited to addition combin.		
2A	Reading: 1 to 1000 Counting: by 3's to 30 by 4's to 40 by 5's to 50	Any 2 Nos. 1 to 10	Minuend to 20. Larger No. of combinations because of more additions.		
2B	Reading: 1 to 1000 Roman to XII Counting: 2's to 100 3's to 100 4's to 100 5's to 100	45 Combinations. Add any No. 1 to 6 to any No. 1 to 100 in series by 10	Minuend to 100		
3A	Reading: 1 to 10,000 Roman to L Counting: 6's } to 100 8's } 2's } begin at any 3's } digit	Any No. 1 to 10 to any No. 1 to 100 in series by 10. Sums up to 10,000.	Subtrahend any No. 1 to 9. Numbers of 4 orders.	Tables through 6×12. Mul. by 1 or 2 orders.	Find factors when multiples are given. Divide by any No. up to 6.
5A	Three Periods. Relation of Orders of Integers; fractions and mixed numbers.	Drill in all forms to develop accuracy and speed.	Drill in all forms to develop accuracy and speed	Drill in all forms to develop accuracy and speed	Drill in all forms to develop accuracy and speed
6A	All forms and variety.	Drill in all forms to develop accuracy and speed.	Drill in all forms to develop accuracy and speed	Drill in all forms to develop accuracy and speed	Drill in all forms to develop accuracy and speed.
6B	All forms and variety.	Drill in all forms to develop accuracy and speed.	Drill in all forms to develop accuracy and speed.	Drill in all forms to develop accuracy and speed	Drill in all forms to develop accuracy and speed.

IN ARITHMETIC, SHOWING SPIRAL DEVELOPMENT

TABLES, WEIGHTS AND MEASURES.	FRACTIONS.	PROBLEMS.	MENSURATION.	BUSINESS FORMS.	DECIMALS.	PERCENTAGE.
Measure length and contents by indef. units, hand, span, etc.		By counting.	Measure length by indef. units.			
Same plus definite units, c, 5c, d., $, foot, quart.	Halves through objects.	By counting.	Measure lengths by ft., and apply other units.			
Same as above plus inch, pt., gal., quarter, and half dollar.	Halves and quarters through objects.	By adding.	Same, with more varied units.			
Same as above plus hrs. of day, days of week, time by clock.	Same plus eighths through objects.	By adding	Same, with more varied units.			
Same as above plus dos., lb., ½ and ¼ lb., min., hr., day, mo., yr.	Same as above plus thirds, sixths, fifths. ———— ⅓ of 20 ⅓ of 30 ⅕ of 50	Any problem of one operation.	Same with more varied units.			
	Proper and improper. Reduction H. and L. terms. Red. mixed No. to frac. Cancellation, add and sub. mixed Nos. mul. by integer and by frac.	Case I, II, III applied to fractions.	Use tables known in problems. Areas.	Bills and receipted bills.		
Reduction asc. and desc. in avoir., time, liquid, dry, tables. Difference in dates.	+, −, ×, ÷ of frac. and mixed Nos. Complex fractions. Laws governing num. and denom.			Same for services rendered.	Red. from common to dec. and vice versa. +, −, ×, ÷	As dec. of 2 orders. % equiv. of frac. Case I applied to profit and loss and commission.
Red. asc. and desc. in tables of lin., sq., and cube measure. Review last grade.	All forms of drill in fundamental operations.	All 3 cases. Use of com. and dec. frac. as percent equiv. Solution shown by signs and equations		Same plus checks and personal accounts.	All forms of drill in the fundamental operations.	Cases II and III, related to business transactions. Interest for yrs. and months.

oughness. In the spiral course, fractions may be taught in three or four grades; only the simplest concepts of fractions are developed early in the school course, and term by term the necessary facts and generalizations are introduced in a graded development which parallels the child's growing mental grasp. To attain these reputed advantages, curricula in arithmetic were made severely spiral in their organization.

Limitations of Spiral Courses in Arithmetic.—An extremely spiral course often suffers from the defects of its own virtues. (1) Reviews and drills, within bounds, are advantages but when rational limits are transcended, they become stultifying repetitions. As the pupil goes from grade to grade in an excessively spiral course, he loses interest in the round of monotonous repetitions. It seems to him as if he were acquiring nothing new in the successive grades. An examination of many of our six and eight textbook series in arithmetic often discloses repetitions in the books assigned for two successive grades that make them almost identical in scope.

(2) A second weakness of such a development is the apparent lack of unity. Teachers see nothing distinctive in each grade. They teach almost the same round of subjects and see no need of unifying the various topics. Teachers in fifth and sixth year classes often teach numeration and notation in the early part of the term, drill in fundamentals the second part, in fractions the third part, etc., and thus revert to the system of exclusive attention to one topic, not for a whole term but for part of the grade. Other teachers try to cover the assortment of topics each week. On Monday, notation and numeration and fundamental processes of whole numbers are taught and drilled; on Tuesday, fractions monopolize the lesson; on Wednesday, decimals; on Thursday, problem work; Friday is devoted to the grand review of the week's experience. Obviously, neither plan

possesses pedagogical merit. Teachers find it exceedingly difficult to plan the term's work when the grade assignment is so rich and varied.

(3) A third element of weakness often results from the much emphasized correlation of the various topics of arithmetic. Interrelations do not always clarify. To the student who has a bird's-eye view of a subject and has an elementary knowledge of it, interrelations give new viewpoints and deeper comprehensions. But to the pupil who is only threading his way through the subjects, these very interrelations may become stumbling blocks and sources of confusion. Teachers often give evidence in support of this fact; they tell of befogged ideas and misinterpretations in arithmetic due to premature correlation.

(4) Supervisors of elementary schools often complain bitterly against extreme spiral courses in arithmetic, for they make it impossible to fix responsibility for poor instructional results. When every teacher teaches almost everything in arithmetic, who can be blamed for specific shortcomings? Teachers are prone to take results as they come, and find comfort in the fact that their colleagues, who will have their charges the following term when the pupils are older, will be able to bring the class to that degree of skill, speed and accuracy which they themselves do not seem capable of developing. This exceeding faith in the possibilities of fellow workers may be complimentary, but it discourages earnest effort on the part of some teachers.

The Modified Spiral Development.—Theoretical analyses and practical experience point to the need of evolving a modified spiral course of study in which the spiral characteristic will be basic throughout the course, but in each grade certain branches of arithmetic will be so stressed as to mark them the distinctive contribution of that semester. There is every reason for retaining the spiral system in the earlier grades. But beginning with the middle of the ele-

mentary grades, opportunity must be given to focalize attention on fewer topics and processes so that drills may be more intensive, and the resulting skill of a higher order. The severely spiral course presents a merry-go-round of topics in each grade and invites a feather-duster treatment of each. In the modified spiral system, it is possible to subject each topic to careful study so that thoroughness of comprehension and permanence of knowledge may result.

THE TIME TO BE DEVOTED TO ARITHMETIC

No Set Time Yet Determined.—A problem which seems of vital significance to the maker of courses of study is the allotment of the approximately correct proportional time to arithmetic. The U. S. Bureau of Education, making its computation on the basis of courses of study used in fifty of our leading cities, finds that about 15¼ per cent of the time devoted to all subjects in the curriculum is given to arithmetic. This averages about 45¾ minutes per day, assuming the regular five-hour day. But when we recall the fact that some school systems have no formal work in the first year; that a few have none in the second year; that in school systems where formal work begins in the first year, the daily period is limited to about 20 minutes or less—then we realize that to maintain an average of 45¾ minutes for all grades, it is necessary to devote much more than an hour a day in the highest grades.

Relation of Time to Proficiency.—A comparison of the time allotted to arithmetic in courses of study used in typical school systems reveals only one interesting fact—the widest range of variations. Obviously there are no scientific data to guide us in determining approximately what part of the school time to devote to arithmetic. *A priori*, it would seem that the more time given to the teaching of arithmetic, the greater must be the proficiency developed by the pupils.

But investigations made under experimental conditions do not warrant such an inference. They seem to indicate no relationship between time expenditure and the results of instruction. If we were to select at random a school from a list of schools that give more than average time to arithmetic, the chances are about equal that we would have chosen a school where the work is below standard, and, conversely, if we were to select a school from among those that devote less than average time to this subject, we would probably find the children doing superior work in it. It was found that schools that attained a rating of 80 per cent in arithmetic and those that scored only 25 per cent in the same tests gave the same time to this subject. The accompanying table summarizes in part the result of C. W. Stone's investigations: [1]

The school system that stood 1st out of 37 school systems gave 12 per cent of its time to arithmetic.
The school system that stood 5th out of 37 school systems gave 14 per cent of its time to arithmetic.
The school system that stood 7th out of 37 school systems gave 17 per cent of its time to arithmetic.
The school system that stood 13th out of 37 school systems gave 11 per cent of its time to arithmetic.
The school system that stood 13th out of 37 school systems gave 13 per cent of its time to arithmetic.
The school system that stood 15th out of 37 school systems gave 19 per cent of its time to arithmetic.
The school system that stood 22d out of 37 school systems gave 20 per cent of its time to arithmetic.

It seems obvious that time alone is not the determining factor in making for proficiency in arithmetic, nor is it as important when compared to other influences that make

[1] C. W. Stone, "Arithmetical Abilities and Some Factors Determining Them," Col. Univ. Cont. to Ed., 19 (1908).

for development of power, as was traditionally supposed. The modes of instruction, the character of supervision, and the organization of the course of study are more telling elements to be considered in trying to bring children to a desired degree of proficiency.

When Shall We Begin the Teaching of Arithmetic?—*The Plea for Late Introduction.*—An ever-pressing question which is asked in the teaching of all school subjects is the time for their introduction. Many parents and educators are urging the postponement of the teaching of formal arithmetic to the second or to the third year. The reasons are the oft-repeated ones: (1) children would probably learn the initial parts of the subject more rapidly if they were taught them when the mind is more mature; (2) the later the subject is introduced, the more clearly does the child perceive its need, and hence greater attention is assured because of rational motivation; (3) the physical welfare of the child demands postponement of formal book work and an increase in active education; (4) many of the basic number facts would be learned incidentally and time will not be lost in the long run. These are only a few of the set arguments for the exclusion of formal instruction in arithmetic from the first year or two of the elementary curriculum.

The Case for Arithmetic at the Beginning of the Course.—But there is a growing conviction that incidental teaching resolves itself into no teaching and precarious results, for teachers are prone to neglect necessary natural drills in informal exercises. The earliest grades are devoted to inculcating necessary habits in arithmetic, not to securing clear comprehension. It seems wise to introduce addition and multiplication tables as early as the child will accept them, so that necessary habits will be developed in the fundamental processes. For a large part of the school population, the educational period is incredibly short and nec-

essary number facts must be acquired and skill developed from the very earliest grades. And finally, it is urged that teachers and supervisors give evidence that the normal child of seven, taught by proper methods, understands the basic concepts of arithmetic and develops skill in their use.

Early Instruction in Arithmetic Need Not Be Formalized.—But it must not be assumed that because arithmetic is made an integral part of the work of the first two years that it is advisable to return to the deadening grind of the mechanical drills in additive combinations or in multiplication tables. Our psychology forbids such procedure; modern methodology condemns such practice. Although the work of the earliest grades is definitely and formally prescribed, it may be enlivened in many ways. Every number fact or process must be motivated by showing its social worth or by making it appeal to the instincts of the child; throughout the development of the early lessons, the inductive method must prevail so that the child is the active discoverer of knowledge; every drill must be presented through some form of play activity and must appeal to the ultrasensitive senses of the child. Arithmetic so taught is vibrant with life and interest.

Arithmetic in the Kindergarten.—It seems to be a general belief among kindergartners that children of five and six are usually too young to grasp number facts aside from the use they make of them in activities incidental to games and manual work. Although no attempt should be made to give formal instruction to these children, certain mathematical knowledge cannot be kept from them. Their work and their play necessitate counting to twenty and even to thirty, the recognition of certain geometric forms, recognizing groups of five or six objects, adding, subtracting, multiplying and dividing small quantities, and producing halves and quarters by folding paper or other objective

devices. But all these arithmetical facts should be acquired, not by consciously setting out to add these definite ideas to the child's knowledge stock, but incidentally, in the round of pleasurable activities, while the mind is bent on more attractive ends.

CHAPTER III

GENERAL PRINCIPLES GOVERNING THE TEACHING OF ARITHMETIC

The Dissatisfaction with Results of Instruction in Arithmetic.—There are few complaints against current education that come with greater frequency than those directed by business men who find the school graduate unable to measure up in arithmetic to the demands of commercial life. It is charged that he, the graduate, is inaccurate, unduly slow in his calculations, and unable to meet a new arithmetical situation, however slightly it is changed from the type. School superintendents have undertaken to systematize the demands in arithmetical knowledge, which business firms make of the unspecialized workers. Invariably these demands are very simple when compared to the standards set up by elementary schools. Dr. McAndrew, Associate Superintendent of Schools, of New York City, selected two firms in the Borough of Brooklyn, which employ the greatest number of school graduates. The managers of these firms were asked to give specimen tasks most commonly required by school graduates in their employ. Their requirements in arithmetic are summed up in the two types which follow:

1. Addition: A sum of about this difficulty.

$$\begin{array}{r} \$1.58 \\ .65 \\ .49 \\ 2.98 \\ 10.77 \\ 1.38 \\ 3.10 \\ \hline \end{array}$$

2. Penmanship, multiplication, fractions, addition, filling out a sales slip about like this one from dictation, computing the values, and adding the amounts:

Bought by Henry Wise,
649 Classon Ave.,
Brooklyn.

Sales No. 163	Carrier 15	Charge Ch	Date 5/16/15

Quantity	Items	Amount
$2\frac{1}{4}$	White lace @ 57c	
3	Spools white silk @ 10c	
$1\frac{1}{4}$	White silk @ 93c	
		Amount

The other tasks are not given because they do not bear on arithmetic. These tests were given to the pupils of the graduating classes of the Brooklyn division with the following results:

Test No. 1	Simple Additions
Number Tested	1023
Number Right	778
Per cent Right	76%
Poorest Class Record	68%
Best Class Record	91%

Test No. 2. Penmanship, multiplication, fractions, addition.

Number Tested	962
Number Right	539
Per cent Right	66%
Poorest Class Record	42%
Best Class Record	92%

Without going into the details of marking, evaluating answers, norms used, etc., this may be regarded as a typical result which gives point to the prevalent dissatisfaction with school work. Teachers spend hours each week for eight years striving for attainments in arithmetic that far exceed the modest requirements set up by these employers of school graduates. What a gulf between the actual achievement and the standards striven for! How can we explain the conditions which produce such waste in teaching? Many factors may be enumerated, chief among which are the following: Absence of specific aims in the course of study; failure to introduce a common standard of measuring progress in all grades; lack of knowledge of the business man's demands in arithmetic; differing views as to the worth of arithmetic, which produce a variety of purposes that govern instruction; different views as to what subject matter to select to fulfill the requirements of the course of study; teachers poorly prepared in the subject matter and in methods of instruction in arithmetic; making arithmetic a fetish of elementary education and thus wasting time by attempting to attain standards unnecessarily high. It is evident that much can be done through more rational courses of study and improved methods of teaching. The preceding chapter was given to a study of curricula in arithmetic and their organization along practical lines. It is the function of this and the succeeding chapter to formulate guiding principles for more effective instruction.

ARITHMETIC MUST BE HUMANIZED

The first basic principle which must obtain in the teaching of arithmetic requires the establishment of a most intimate relationship between the problems of life and the problems of arithmetic. Changing the order of topics in arithmetic, eliminating certain obsolete forms, emphasizing the heuristic method or any other procedure—these are insignificant considerations when compared to the need of a social attitude in arithmetic. Unless the problems of arithmetic are real, and their solution seems urgent, the subject will always lack human significance. An examination of problems taken from current tests in arithmetic, from manuals of methods of teaching, and from courses of study, shows clearly how far we are from this standard. Thus, one author asks: "A, B and C eat eight loaves of bread, each the same amount; A furnishes 3 loaves and B 5 loaves. C pays 24 cents for what he eats. how much should A receive? Make an analytic diagram." Later in the text the same author is troubled by another situation: "A is 6 feet tall. It is proposed to make his statue 12 feet tall. A's little finger is 3 inches long. To paint a statue of A's size costs $2. The weight of a statue of A's size is 1000 lbs. What will be the length of the little finger of the statue? What will it cost to paint the statue? What will be the weight of the statue?" Lest the questions go unanswered, this author presents the solution: "A's height has been multiplied by 2; multiplying a linear part by a number, multiplies the linear part by that number, multiplies a surface part by the square of that number, and multiplies a solid part by the cube of that number; 2 inches, the length of the little finger, must be multiplied by 2; $2, the cost of painting a statue of A's size, must be multiplied by the square of 2; 1000 lbs., the weight of the statue, must be multiplied by the cube of 2." The problem and its solution are offered as models

to young teachers in a book on the teaching of arithmetic.

Problems as far removed from life may be selected from other books just as well known as the one from which the above quotations were taken. Further illustrations are cited in the chapter devoted to a study of problems. It is evident that instructional reform in arithmetic must reach deeper than mere rearrangement of some topics or elimination of others. Unless the problems and the situations are real, arithmetic will continue to be devoid of human touch.

MOTIVATION IN ARITHMETIC

Value of Motivation in Arithmetic.—The best safeguard against teaching what is socially useless in arithmetic is to make an earnest endeavor to convince the pupil of the need he has for the knowledge we are about to present. It will not suffice to justify the lesson to ourselves. We must try to interpret the topic to the child in terms of need and use in his life. If teachers and textbook writers tried to motivate the problems they give to children, they would, doubtless, eliminate much that they now regard as important. Once social worth is shown, we are reasonably certain that the child is actuated by real motive; that attention will cheerfully be given; that interest of a dynamic nature is aroused; that self-activity will flow in plenty, and that the class will coöperate throughout the entire lesson.

How to Motivate Arithmetic?—But the vital question which the teacher asks at this point of our discussion is, naturally: "How can the subject matter of the course of study in arithmetic be motivated?" The variety of methods of motivation in arithmetic is rich and simple of attainment in most grades. To these we must now turn.

1. Arithmetic can be motivated by showing the pupil the *need* he has for a specific process or set of facts. In teaching business forms, bills, receipts, checks, single entry

bookkeeping—topics so intimately related to the child's experience—the mere mention of the aim of any of these lessons will serve as a motive. A lesson on multiplication by fractions can be motivated by making one item of a bill, " $1\frac{1}{2}$ yd. at $2.40 " and leading the pupils to foreshadow the aim of the lesson—" to learn how to multiply with fractions." Similarly, a lesson on division by a decimal can be motivated by giving the children a practical problem, the solution of which is found by dividing by the decimal .6. The children do not proceed very far before they inform the teacher that they do not know how to manipulate the decimal point. The pupils are then asked " What must I teach you now? " and the children state the aim of the succeeding lesson. These are typical illustrations of a host of similar situations in arithmetic in which motivation can be achieved by showing that the new process is necessary in the pupil's life.

2. A second means of arousing rational motive is to *relate new experience to an object or an end in which the child is directly interested.* A pupil who is not interested in arithmetic but is fascinated by work in manual and vocational exercises may be attracted to arithmetic by deriving all of its problems and processes from the activities that he likes. Such a pupil can be taught addition, subtraction, multiplication and division of fractions, reductions and various manipulations of denominate numbers by relating all these arithmetical processes to manual and vocational exercises. In making out the price tag for an article of furniture just completed, the child must add a per cent profit to the cost of material and labor. Here we have the motive for teaching this child the initial lesson in percentage. In similar manner a child's interest in athletics may be utilized as a motive for arithmetical purposes. Four schools in athletic competition have thus far the following records:

School I.....	Won 4	Lost 3
School II.....	5	2
School III.....	5	3
School IV.....	4	1

These figures can be made the basis of an interesting series of mental arithmetic examples and can be used to motivate teaching of Case II, in percentage, by asking, "How shall we compute the per cent of games won by each?" So, too, interest in physics, geography, or cooking can be made the means of stirring motive and arousing a desire for the mastery of certain arithmetical experiences.

3. A simpler means of attaining the same end, motivation, is to *appeal to any one of the child's impelling instincts.* Through the child's inherent love for play and games, early number work is made the source of great pleasure. Many of the mechanical processes of arithmetic are made natural by appealing to the puzzle instinct, the instinct of curiosity, the urge to overcome obstacles the conquest of which seems to be within our capacities. Children may even find reducing fractions to lowest terms and discovering least common multiples and greatest common factors no stupid processes when these are presented in the light of puzzles which challenge their ingenuity. Adults often spend hours in attempting to solve puzzles in newspapers and in advertisements. When all other attempts at motivation fail, an appeal to an instinct will often prove not only successful, but also a potent spur to intense activity.

METHODS AND PROCEDURES IN THE TEACHING OF ARITHMETIC

The Variety of Methods.—Many systems or methods of teaching arithmetic have been evolved. A tabular arrangement of these methods will afford an easy summary.

1. Inductive v. deductive method.
2. Analytic v. synthetic method.

3. Heuristic method.
4. Laboratory method.
5. Textbook method.

Each of these has its own advantages over the others, but none is so perfected as to be capable of exclusive and general application. An eclectic method formed by a combination of all of these is not always possible nor practical in the teaching of arithmetic. The peculiar merits of any one method often make it the best procedure in a specific case and the least effective mode in another. We must, therefore, study each of the common methods of teaching arithmetic to discover its aim, its procedure and the situations that make its use, singly or in combination with another, most advisable.

1. Inductive v. Deductive Method.—*Chief Characteristics of Induction Summarized.*—The writer assumes that the student of special methods of teaching arithmetic knows that induction is that form of reasoning in which a general law is derived from a study of particular objects or specific processes; that the five formal steps of the Herbartian recitation or the Method-Whole follow a severely inductive development; that induction is the method which leads the pupil to "discover" knowledge; that the law, rule or definition formulated by the child in inductive lessons is the summation of all the child has gathered in the presentation; that induction demands active coöperation on the part of the pupils throughout the lesson. The reader will find many illustrations of inductive teaching in the chapters that deal with fractions, decimals, percentage and mensuration, and in the discussion of the heuristic method in this chapter.

Is Induction in Arithmetic True Induction?—Strictly speaking, pure induction is seldom found in the teaching of arithmetic. In all induction in life, the generalization that is evolved is regarded as a tentative conclusion, a temporary

hypothesis, the validity of which is tested by application or the content of which is modified or enriched through further experience. The law obtained in an arithmetic lesson is final because the teacher selected only those individual instances which point unmistakably to it.

Induction through Typical Instances.—At times the law is evolved from a typical instance and not from a study of a number of specific individual cases. Thus, the general procedure for multiplying by any number of two orders, e.g., by 18, is derived from the procedure followed in multiplying by some *one* number of two orders, or the rule for the division by a decimal may be deduced from the specific example which requires division by some *one* decimal fraction.

Place of Induction in the Teaching of Arithmetic.—Despite the incompleteness and the unscientific character of these inductions, they are, nevertheless, emphasized in the teaching of arithmetic because they follow the general mode of thought in induction. All inductive teaching concerns itself with the development of mental power, mental habits and modes of thought, rather than with the mere acquisition of fact. The inductive method must, therefore, be the predominating method in the teaching of arithmetic.

Characteristics of Deduction Summarized.—It is evident that deductive teaching which begins by positing the law, and then seeks to read meaning into it by attempting to solve specific examples, lacks the instructional elements of strength that characterize induction. In deduction the child does not discover laws, but verifies them; he does not develop a mode of thought, but acquires well-established laws and facts. The lesson in division, given in Chapter VIII, is an illustration of a typical deductive lesson and shows that deductive teaching is not necessarily authoritative instruction imposed on the child without appeal to thought or motive.

Place of Deduction in Arithmetic.—There are conditions in arithmetic when deductive teaching is not only advisable, but almost imperative. Chief among these we must include the following:

(1) When the Generalization is Too Difficult.—There are laws in arithmetic that require powers of analysis and synthesis beyond the meager mental resources of the child. To attempt an inductive development is to produce confusion and discouragement in the child. For this reason we must teach deductively division by a number of two orders, multiplication by a number of three orders, and subtraction with numbers in the subtrahend larger than numbers in the corresponding order of the minuend.

(2) When the Generalization is too Long.—A law can be derived inductively only when the individual cases presented are simple enough to enable children to compare them, to abstract underlying likenesses and characteristics, and then to weld them into a synthetic concept. Obviously, we court confusion when we attempt to develop the method of finding the least common denominator or the greatest common divisor, or the method of dividing by a number of two or more orders. The elements in these procedures are so numerous that no child can maintain enough control over them to enable him to formulate the generalization. Experience teaches that a deductive lesson is wiser, for it makes for economy of time and clarity of comprehension.

(3) When Legitimately Pressed for Time.—There are conditions in the course of class instruction which make it necessary to achieve a definite end in an unusually short period of time. Thus, a teacher in a given grade must teach the next progressive step in denominate numbers or in decimals and finds that the class, for reasons which cannot readily be ascertained, lacks the necessary apperceptive basis which should have been acquired in an earlier grade. It is

evident that the work of the lower grade must be made up. The exigency of such a situation makes it advisable to follow a deductive procedure. Teachers of upper grades of the elementary course, charged with a review of all preceding work, often find that necessary processes have been forgotten by the class. To accomplish the work of review and reteach such parts of the subject as are necessary, a deductive development is often a matter of absolute necessity. It must not be assumed that mere saving of time is the proper justification for a system of instruction that is excessively deductive. There are values in the inductive lesson that more than compensate for the extra expenditure of time which it necessitates. Only when the time cost far outweighs this gain of induction are we justified in giving way to deductive teaching.

(4) When a Topic Assigned is of Little Social Worth to the Child.—Teachers must always scrutinize the course of study and seek to harmonize it with the needs of the specific class to be taught. When a course of study is so rigid in its prescriptions as to make impossible this adjustment, it is wise to teach deductively such parts of the subjects as will, in all probability, play no part in the lives of the children. An illustration may avoid misunderstanding of this suggestion. A teacher of an eighth year class in one of our city schools finds that she must teach the area of a circle to a class of girls whose education, at best, will not extend beyond the elementary school. It is evident that the formula, πr^2, will contribute no material aid in the adjustments which these girls must make immediately after graduation. To teach the area of a circle inductively will take an entire period and leave no time for application; to teach it deductively requires about one-fifth of the period and leaves considerable time for application of this formula and for drill on calculations of greater worth to these particular children. It is futile to argue

that a course of study which imposes such a topic and which is so lacking in flexibility is pedagogical tyranny. The teacher is confronted by a real condition—the course of study must be carried out; the deductive method in such cases minimizes the limitations of the course of study and helps the teacher to square pedagogical practice with professional ideals.

(5) WHEN A TEXTBOOK LESSON IS TO BE GIVEN.—The deductive lesson must be followed when children are given a lesson in arithmetic from the book, in the endeavor to teach them how to study the subject without the aid of teacher or of parent. While the methods of study may be developed inductively, the subject matter of arithmetic will naturally be acquired deductively. Detailed explanation of such lessons, together with appropriate illustrations, will be given under a later topic in this chapter.

From the enumeration of the conditions that make advisable the deductive method, it is obvious that it is not, in the nature of the case, an unusual instructional procedure, and it must not be assumed that deductive teaching is mechanical teaching, authoritative in spirit and characterized by pure memory appeal. The deductive lesson has a rationale and a development as logical and as thought provoking as most inductive instruction. For illustration, the student is again referred to a later discussion—the deductive recitation, in Chapter V.

Supplementary Relation between Induction and Deduction.—From the characteristics of the inductive and the deductive modes, it is readily seen that each of them has compensating values for the limitations of the other. Inductive teaching is absolutely dependent upon deduction, for when it is unsupplemented by deduction it lacks verification and application; exclusive deductive teaching emphasizes facts at the cost of method and fails to teach the characteristic mode of reasoning in arithmetic.

2. Analysis and Synthesis in Arithmetic.—*Meaning of These Terms.*—Analysis and synthesis are terms often misused in the teaching of arithmetic. Students take the elementary conceptions of these terms and therefore think that "synthesis may be defined as a process of putting together, and analysis as a process of taking apart." These definitions are quoted from a book on methods of teaching arithmetic and reveal the crude and erroneous conceptions that have gained currency. Analysis is more than a mere taking apart; it is a separation according to function or according to a principle of classification. So, too, synthesis means assembling elements by inter-relating their functions. To pour the liquid in a receptacle into two or three test-tubes is no act of analysis. To separate the liquid into its two or three chemical elements is an act of analysis. The machinist who puts wheels, shafts and bolts together to produce a machine is synthesizing, for the place given to each part in the organization of the whole is determined by the function it performs. The automobile that is wrecked in a collision is not analyzed; it is smashed. Nor is the act of gathering up the broken pieces one of synthesis.

The Terms Illustrated.—Analysis and synthesis are modes of organizing data and reasons in arithmetic. To teach a child the number eight by counting, one, two, three, four, five, six, seven, eight, or by adding one to seven, is an act of synthesis. To teach *eight* by beginning with eight children and having a pupil arrange them in single file to discover that eight *one's* make eight, in double file to learn that four *two's* make eight, in groups of four to find that two *four's* make eight—these constitute an analytical lesson. The teaching of geometry which begins by positing a proposition and then proceeds to put together known laws, axioms, and propositions to build up to the proposition that introduces the lesson is another illustration of synthetic

reasoning. A typical analytical solution of the following problem is offered below:

A foundry reduces salaries by 8 per cent and thus saves an additional $3248.00 per month. What was the amount of the pay roll before reduction?

8 per cent of salary is reduced.
$3248.00 is the sum reduced.

$\therefore$ 8% of salary = $3248
1% of salary = $\frac{1}{8}$ of $3248 = $406
100% of salary = $40600

Pedagogical Importance of Analysis and Synthesis in Arithmetic.—From the illustrations cited above it can be seen that mathematical synthesis proceeds from the known to the related unknown, while analysis manipulates unknown quantities to discover their worth. Mathematical analysis is, therefore, a method of discovery in which the child acquires a method of mathematical reasoning. Mathematical synthesis is logical explanation, but does not disclose the method of arriving at a new conclusion in mathematics. In mathematical analysis the child is mentally active, while in following mathematical synthesis the child is comparatively less self-expressive. The dominant mode of teaching must therefore be analytical if the mathematics of the elementary school is to exert, in full measure, its disciplinary power upon the developing mind of the child.

3. The Heuristic Method.—*Explanation.*—The heuristic method is more commonly known as the development method and refers to that mode of teaching in which the child is led to rediscover the socially necessary knowledge that the race has gathered. It regards the pupil's mind as concentrated potential energy eager to burst its confines in its endeavor to attain necessary ends. Its aim is not the giving of fact, but the development of power through self-expression.

Illustrations of the Heuristic Method.—The characteristics and the merits of the heuristic method may be made clear through its application to actual instructional situations. In teaching children to subtract 37 from 62, six dimes, and two cents are shown. It is now necessary to pay a debt of 37 cents, i.e., of three dimes and seven cents. It is a simple matter to give my creditor three of the six dimes, but how give seven cents when I have only two? The children are now confronted by a problem. What solution can they suggest? A few judicious questions and suggestions lead them to see the light. Have I enough money? Must ten cents always be in the form of a small silver coin? How many cents can I receive for the dime? If I change a dime into cents, how many cents will I have? The conversion is made and the children now see that they not only have five dimes, but twelve cents—denominations that make it possible to pay my debt of three dimes and seven cents. A few more similar problems are solved and the children discover a way of subtracting numbers of two orders where the digit in the subtrahend is larger than the digit in the minuend.

In teaching division by decimals the heuristic method stirs pupils to similar active coöperation. A problem is given, in which it is required to divide 24.36 by 6. No difficulty is experienced, for the children know this calculation. A second problem is now given in which it becomes necessary to divide 24.36 by .6, not 6 a whole number. Most children begin the division only to stop as they realize that the decimal point of .6 is a troublesome element. The teacher elicits from the children that if it were possible to eliminate the decimal point of the divisor they would readily find the answer. The problem now confronting the class is how to bring about a change in the divisor from .6 to 6. The teacher has recourse to facts known by the children and asks the class to convert $1 \div 2$, $2 \div 3$, $3 \div 4$,

$7 \div 10$, $15 \div 6$, into fractions. The board work presents the following appearance:

$$1 \div 2 = \tfrac{1}{2} \qquad 7 \div 10 = \tfrac{7}{10}$$
$$2 \div 3 = \tfrac{2}{3} \qquad 15 \div 6 = \tfrac{15}{6}$$
$$3 \div 4 = \tfrac{3}{4}$$

The class is now asked to convert $24.36 \div .6$ into fractional form and the answer $\frac{24.36}{.6}$ is readily obtained. The basic problem "how to turn .6 into 6," is now recalled and the teacher asks: "What do I do to the denominator when I change $\frac{24.36}{.6}$ into $\frac{24.36}{6.}$?" When the answer, "It is multiplied by 10," is obtained, the children are asked: "If I multiply the denominator by 10, what must I do to the numerator?" "Have I now changed the value of the fraction?" The work on the board now shows:

$$24.36 \div .6 = \frac{24.36}{.6} \times \frac{10}{10} = \frac{243.6}{6} = 6.)\underline{243.6}$$

In a similar way children are led to recognize the new problem involved in dividing 2.436 by .06 and to discover the mode of solution. After a few similar cases are worked out, the pupils are now ready to formulate the law for dividing by a decimal.

Cautions and Suggestions in the Use of the Heuristic Method.—There are common misconceptions concerning the heuristic method which destroy its value as a mode of teaching. To these we must now turn.

1. *Inductive and Analytic-Synthetic in Character.*—The heuristic method usually follows an inductive rather than a deductive unfoldment and for that reason is analytical in its initial stages but toward the end of the lesson when the new law is discovered and formulated a synthetic process is employed. It is not absolutely true that the heuris-

tic method is always inductive. Children may be told that climate is determined, among other things, by latitude, altitude, distance from a large body of water, prevailing winds, ocean currents and surrounding topography. The problem before the class is to make intelligent inferences concerning the climate of England. These laws of climate are regarded as tentative and are verified by a comparison with the facts found in the textbook. Here, therefore, is a deductive lesson, but one in which the problem was developed by the children. In arithmetic, however, heuristic teaching is generally inductive teaching.

2. *Time and Cost of Discovering Knowledge Must Be Considered.*—We must ever be mindful of the fact that in the heuristic method, knowledge must not be discovered by the children regardless of time, effort and ultimate purpose. The price in time and effort that the race paid for its knowledge must not be repaid by each succeeding generation. Teachers must therefore evaluate the arithmetical experiences they teach in terms of time and effort, and decide whether the fact is worth the cost. Children can be taught reduction ascending heuristically, but we may question the wisdom of spending the necessary time and effort on a topic that plays so little part in the lives of most children.

3. *Heuristic Teaching is Directed Teaching.*—It must not be erroneously conceived that heuristic teaching entails blind, wasteful effort, a series of trials and errors with ultimate accidental success. It is the teacher's duty to lead the child along the correct path and guard, by questions and suggestions, against pitfalls in reasoning and against unwarranted conclusions. The heuristic method is therefore not a method of learning devoid of teaching, but rather a mode of instruction which regards teaching as the art of systematically arousing and directing the self-activity of the pupil until a desired end is attained.

Limitations of the Heuristic Method.—The cautions that

were discussed in the use of the heuristic method show clearly that it is not without its limitations. (1) Its cost in time is often a serious factor, although not one which precludes its use to any great extent. Since growth in mental power rather than in acquisition of facts is the desideratum, the course of study can be made less extensive and the time thus saved can be invested in pedagogical teaching. (2) The heuristic method is not suited to all children. Those lacking in mathematical power are often pushed through a series of explanatory steps, and imitatively repeat "therefore's" that are imposed upon them by teachers who hope to lead children to rediscover the knowledge of the race. An artificiality is thus produced which violates all the principles of psychologized pedagogy. (3) The heuristic method often encourages wandering from the purpose of the lesson and from the line of thought designed to lead to the desired conclusion. In the hands of an unskilled teacher, questions are clumsy, irrelevancies are not excluded, every passing suggestion is followed, and the heuristic method gives way towards the end of the lesson to a telling method. All these limitations are not beyond remedy and they argue for intelligent application of the method rather than for its elimination

Values of the Heuristic Method.—It remains for us to sum up the advantages of the method which were seen in the illustrations. (1) The heuristic method stirs self-activity and makes the child an active participant in the development of knowledge. The entire lesson is now permeated with that personal warmth which is felt for everything that is the result of one's own effort. In the illustration cited above, the law for dividing by a decimal is the child's own, the result of his attention, his insight, and his formulation. (2) Such teaching brings better comprehension and (3) more lasting impression. (4) The heuristic method always rouses a feeling of confidence in a child. Step by

step he followed the line of thought, held to it at every turn, and finally summed up its essence. The child realizes that he has power and capacity, a feeling which no book method can give. (5) A final worth of the heuristic method is found in its ability to give the child a mode of thinking in arithmetic. This new power helps him to reconstruct the law for division by a decimal, should he forget the process. It also helps him to evolve a solution when new elements enter into this arithmetical experience. The child who is taught to divide 24.36 by .6 or 2.436 by .06 by rule of thumb is at a loss when the example is 24.36 by .006. But when the pupil contributes towards the development of the law he usually has enough insight into the process to meet the difficulty. So, too, in the example $600\overline{)543.6}$; when such a pupil is told to divide, not by 600 but by 6, a mere suggestion will show him that since the divisor is divided by 100 the dividend must undergo similar treatment and the solution is simplified into $6.00\overline{)5.436}$. The time of teaching the original law heuristically is more than compensated for by the insight into number, by the mode of thought that is acquired and by the time that is saved in later teaching.

4. Laboratory Method.—A suggestion for improved and vitalized teaching of arithmetic has in late years assumed the dignity of a " method " and the pretentious title of the laboratory method or the Perry movement. It urges teachers to begin arithmetic in the needs of the pupils' lives, to teach it by appealing to children's instincts, to present the subject psychologically rather than logically, to make arithmetic a phase of geography, physics, history and industry. In essence the laboratory method reduces itself to the problem of supplying an intelligent and a social motive for instruction in arithmetic; the principle of teaching which this method embodies was discussed with greater fullness in the previous study of motivation.

In so far as the laboratory method would make arithmetic solely incidental in other subjects as the need for mathematical solutions arises, it suffers from serious limitations. Such instruction in arithmetic makes no provision for the rigorous drill needed to insure habits of accuracy and speed in the manipulation of fundamentals, nor for the gradual instruction in the elements of arithmetic which will bring the student to those more advanced parts of the subject that can be applied in the solution of real problems. Then, too, it must be remembered, there are certain processes in arithmetic which are technical in their nature and which cannot be applied directly in the solution of a real problem, but which, nevertheless, are *sine qua non* in acquiring such a mastery of numbers as will make this practical application possible. Finding least common multiples, factors, short cuts and learning the decimal notation are a few of these technical elements which, directly, are inapplicable to the solution of real problems, but which, nevertheless, are indirectly the basis of all practical operations in arithmetic. The laboratory method indicates a helpful tendency in arithmetic, for it counsels correlation with the quantitative problems of life, and to that extent relates arithmetic most intimately to the needs and the environment of the pupils.

5. The Textbook Method; Teaching Children to Study Arithmetic.—*The Problem a Vital One.*—The most dynamic conception of teaching, that is, the art of arousing and directing pupils' self-activity toward a definite goal, implies that the pupil is the follower and the teacher the leader. For this reason, an instructional program of exclusive teaching, however efficient that teaching may be, is ultimately enervating, for the student is not taught to select his own goal and to direct his own energies toward its attainment. If the final aim of the teacher is to make himself superfluous, it is obvious that children must be taught

the technique of book study, for the book must ultimately become the source of knowledge in post-school days. The teacher who neglects the use of the class text or who neglects to teach children how to study a new lesson from the book fails in the most vital aspect of instruction.

Illustration of the Use of Textbook in Class Teaching.—A school arithmetic used in many classes introduces its examples and problems in reduction descending of denominate numbers as follows:

Reduction Descending

It often becomes necessary in practical computations to change 3 gal., 2 qt., 1 pt. to pints. The following form is suggested:

gal.	qt.	pt.
3	2	1
×4		

12 qt. + 2 qt. = 14 qt.

×2

28 pt. + 1 pt. = 29 pt.

Explanation.—There are four quarts to a gallon. Hence in 3 gal. there will be 4 times the number of quarts. But it is necessary to add the 2 qt. in the given quantity, hence 3 gal. and 2 qt. are reduced to 14 qt. There are 2 pt. in 1 qt. In 14 qt. there will be 14×2 or 28 pt. But there is 1 pt. in the original quantity, hence 14 qt. and 1 pt. are reduced to 29 pt. *Ans. 29 pt.*

The model solution and explanation are followed by a long list of examples requiring the reduction of denominate numbers from larger to smaller units.

Teachers who make a fetish of developing all lessons in all subjects carefully guard their children from this short and rather formal explanation of reduction descending, and give themselves to the extravagance of an inductive development through the formal steps. This inductive

recitation, because it is logical, well graded, and clear, leads the children to formulate the rule. But in the final analysis the teacher was the leader and the pupils the followers. The greater the merit of the traditional teacher, the simpler is it for the class to follow in his footsteps. But such teaching, which reduces itself to skillful leading, develops mental dependence and discourages self-instruction.

A wiser procedure is less elaborate, less inductive and makes the formal explanation and model solution of the book the basis of the lesson. The teacher asks, " What will 3 gallons, 2 quarts and 1 pint cost at 6 cents a pint? " " What will I pay for 4 bushels, 2 pecks and 4 quarts at 12 cents a quart? " A judicious suggestion or question leads the class to conclude that it is necessary to know the number of pints bought in the first example and the number of quarts in the second. The teacher then asks a pupil to formulate the aim for the day's arithmetic period. The class is then told their textbook offers the solution on page x. All turn to the topic, " Reduction Descending." The teacher then asks: " What would you do with this lesson, if you had no teacher? " With the aid of the class, it is decided that the following steps would be taken:

(1) Find the meaning of " Reduction Descending." Method: (*a*) Use dictionary; (*b*) Examine type of examples.

(2) Attempt to understand explanation given in the book and apply it to the example worked out.

(3) Work out same example without the book. Compare answer.

(4) Work out similar example in same table. Check answer.

(5) Work out similar examples in other tables. Check answers.

These steps must now be taken by the children. They search in the dictionary for *reduction* and *descending*, study the problems on page x and conclude that reduction descending means "changing larger units to smaller ones."

The teacher asks them to prove their definition and the children, examining each example, find that miles must be changed to rods, yards to inches, gallons to pints, etc. In studying the explanation of the solution, children should not be told the meaning of words. These should be found in the dictionary. All the problems that arise must first be subjected to the pupil's most serious efforts, and only after repeated failures should the teacher proffer aid. If the children are ready for this type of example there should be no difficulty in having them master the solution through a study of the book. The lesson suggested is deductive, entails a minimum of guidance on the teacher's part, a maximum of effort on the pupil's part, and teaches the children the art of study

Further Uses of the Textbook in the Teaching of Arithmetic.—The chief instructional values or uses of a good book in arithmetic can readily be summed up under the following heads:

(1) It is an instrument for developing in children habits of study, the art of extracting thought from the page, and a feeling of self-reliance. The foregoing illustration is an attempt to demonstrate this function of the arithmetic book.

(2) It makes for economy of time and effort. A good arithmetic book reduces the need of dictating examples to the class. All that need be done is to mention the page and the number of the exercise.

(3) It offers a collection of exercises and forms of application possessed of greater variety and more careful gradation than most teachers can work out to meet a special teaching need. No slur is implied in this statement. Writers of textbooks of arithmetic are usually specialists of wide experience and their texts are generally the result of careful selection and liberal eliminations.

(4) It is a means of verification. In all inductive les-

sons children should be taught not to accept the laws, the definitions, the rules that they formulate, as final. It is wise to develop in children the attitude of regarding their generalizations and inferences as tentative, to be accepted as final only after these have been compared with the textbook generalizations and applied to the solution of problems.

We see then that while the arithmetic book need not be followed religiously, page by page, it may, nevertheless, at times supplant class instruction. Usually it supplements the teacher's instruction by summing up the laws, definitions, tables and solutions evolved in class. To neglect the textbook in arithmetic is to rob the subject of much of its disciplinary worth for the elementary school.

Qualities of a Good Book in Arithmetic.—The practical teacher realizes at once that not one of these values of the textbook method can be realized unless the arithmetic book is properly organized and developed. The concluding question in the discussion of the textbook method and of teaching children to study must, therefore, be: "What qualities characterize an effective arithmetic book for the elementary grades?"

(1) A good arithmetic book must not include the whole subject in an endeavor to meet the needs of the entire school course, unless it is designed for review work in the last grades. It is evident that the language, the forms of exercise, the explanations and the definitions must change with the progressive stages of the school course. Therefore, no one book can meet the needs of all grades. For this reason the bulky "Complete Arithmetic" of former years has given way to smaller arithmetics written in a series of volumes, each covering a limited portion of the school course. In most school systems it is deemed wise to introduce the book as early as the third or the fourth year, depending on the pupils' language ability.

(2) A satisfactory text in arithmetic is replete with a variety of well-graded exercises and forms of application. The teacher cheerfully assumes the task of instruction, but looks to the book to relieve her of the labor of gathering suitable exercises, examples and forms of application. It is in the application step that most teachers expect the book to render its greatest service.

(3) The forms of application in a good arithmetic book are intensely practical and closely related to the commercial and industrial needs of the community. For this reason it is becoming customary to adapt standard arithmetic books to changing localities.

(4) The problems and exercises in an arithmetic book must be designed for oral as well as written solution.

(5) Whether a good arithmetic book should contain the correct answers to the problems is a question that is much disputed among teachers and supervisors. To give the child the correct answer is one way of enabling him to check his work and test its accuracy. In the event of error, the correct answer gives him a motive for working the problem over again. Many a child has evolved a mode of solving a problem by working toward the correct answer or backwards from it.

(6) Another requisite of an effective book in arithmetic is that it should be devised not solely for the purpose of supplying exercises, but also for the purpose of enabling the teacher to instruct her pupils in the art or studying arithmetic. Most arithmetic books published in very recent years are devoid of all explanatory and expository statements, all definitions and diagrammatic presentations. The authors of these books assume that arithmetic is to be learned only from didactic instruction by the teacher. When one turns to the topic " Insurance " or " Division by Fractions " he finds only a long list of graded and varied examples. Manifestly such books cannot be used in the

class for the purpose of teaching children how to study new lessons without direction by the teacher. A good arithmetic book gives simple and clear explanatory statements of new matters, diagrammatic illustrations of quantitative relations, a complete index, and adds all that is reasonably necessary to enable children to acquire some knowledge from the printed page as well as from the teacher. The modern doctrine of self help must be kept in mind in planning textbooks in arithmetic.

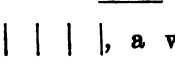
To these demands of a textbook in arithmetic we must add those characteristics that are expected in a good book on any subject. The language must at all stages be suited to the children; the development must, at all points, be psychological rather than logical, so that there will be a vital point of contact between the child and the new topic; the mechanical requisites of paper, print, style and binding must measure up to approved standards.

OBJECTIVE TEACHING IN ARITHMETIC

Lessons from Psychology.—The observer of classroom methods of instruction in arithmetic is forcibly impressed by the unorganized, ill planned and excessive use of objective teaching in the early grades and its thorough elimination from the upper grades. This condition is due to the naive assumption that modern psychology teaches that the young child is in a " sense perception stage " while the older pupils are in the " conceptualization stage." For this reason there is a superfluity of objects, pictures and diagrams at every step in the work of the early grades, and an insistence on the memorization and mouthing of imperfectly understood generalizations in the higher grades. We must realize that modern psychology insists that all primary ideas, all new concepts, must be built up through direct experience with a variety of concrete presentations, irrespective of the age of the student. The pupil, who in the

last two years of his school course is studying the metric system, must be taught these new units and their relations in as objective a manner as the table of liquid or linear measure was taught in the early school years. The newness of the concept, and the previous experience of the pupils, rather than their age, determine the degree of objectiveness that must be incorporated in our methods of teaching.

Qualifications of Effective Objective Teaching.—But unplanned objective presentation or a conglomerate of objects does not make for effective objective teaching. Unless objective demonstrations possess certain qualifications, teachers will find their efforts, along this line, doomed to failure. What are the essentials of effective objective teaching?

1. *Variety of Objective Presentation.*—A serious fault in current objective teaching of arithmetic is the limited range of materials. The child is confronted by the monotonous round of sticks, beads, beans, papers, strings and lentils as he goes from class to class Proper variety can be introduced by: (*a*) *changing objects, materials* and *activities;* using the variety of traditional classroom objects in *games, community activities,* playing at various *occupations,* etc.; (*b*) *changing the sense appealed to* in the lesson; thus children add sounds, steps, hops, as well as the objects they see and handle; (*c*) *changing the combinations* that can be made with the ordinary objects; a child may be taught the combinations in four by using only four sticks. The child is directed to make a box of his four sticks, , Indian tents, , four Indians  , a weather flag, , a chair, , etc. In the course of these new forms and combinations made with the same four sticks, the child is taught that 4 is four

one's, two two's, two and two, three and one, one more than three, two more than two.

2. *The Objects Used in Arithmetical Relations Must Be Part of the Child's Environment.*—To add beads or beans, to subtract lentils, or to manipulate dots in rows and columns, introduces an artificiality into arithmetic that robs it of motive and purpose. We must, as far as possible, make the objective manipulations natural. For this reason, playing at various occupations like store, baker, butcher, etc., practicing community activities, playing games, solving problems based on seating arrangement in the class, etc., constitute more natural and, therefore, more effective objective work. We must ever be mindful of the fact that unnatural objects are as unobjective as mere verbal teaching.

3. *The Object Must Not be a Symbolic Representation.*—The terms objective and concrete, as used by the layman, cannot be applied in teaching without adding a distinctive idea. The mere fact that a thing can be sensed does not make it concrete or objective in teaching. The truth of this contention can readily be seen from an illustration. The following problem was presented to a fifth-year class: "I lost $\frac{1}{5}$ of my money and found that I had \$240 left; how much had I at first?" The teacher decided on a concrete demonstration of the problem by means of the following diagram:

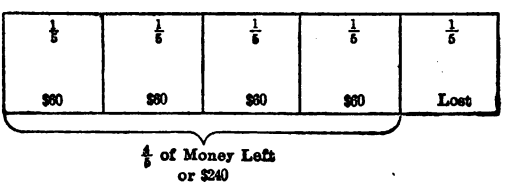
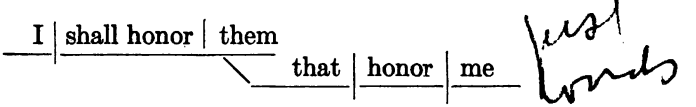


FIG. 4.

Upon closer examination what have we here? A long rectangle *represents* "all the money I had at first." This is

broken up into five-fifths. One small part, a fifth, *represents* "the money I lost," and four of these small boxes *represent* "the money that was left." The conclusion to which the teacher tried to lead the class was: "therefore, four of these boxes represent $240." But in the final analysis this is not a concrete presentation but a *symbolic representation*, for the long rectangle *represents* one quantity, each part *represents* another and the grouping into four-fifths and one-fifth *represents* a quantitative relationship. The diagram above does not make the problem any more concrete than does the diagrammatic representation of the grammatical parts of a sentence. The sentence, "Them that honor me, I shall honor," may be analyzed diagrammatically as follows:



But it is apparent that this analysis by diagram does not make the grammatical relations concrete. Analysis by diagram is a symbolic representation, not a concrete presentation. So, too, in teaching children to write numbers of two orders, e.g., 15, 16, 12, by a device which makes pupils tie sticks into bundles of ten, we are not, strictly speaking, giving a concrete demonstration. To make the first problem concrete it is necessary to make the amount of money very small, say 24 cents, and then evolve the solution by actually using the money. The same type of problem can be made concrete by changing the form to read: "One-third of my ruler is uneven; this makes 4 inches useless; how long is the ruler?" After a number of such problems is solved with actual money and actual rulers, then the teacher shows that whether we say ruler, stick, measure, money, makes no difference, for the mode of solution and the diagram are the same. But to introduce this type of

indirect case by the first example given above and then to represent the quantities and the relations among them by a series of symbols adds to the child's difficulties in trying to understand the new situation, the necessity of learning what the symbols symbolize.

4. *Where Possible, the Objects Used Must be of Proved Worth.*—Experimental pedagogy is now busy testing the relative worth of various forms of objective materials. Evidence gathered under experimental conditions tends to show that disklike and spherical objects bring out the fractional idea much better than linear representation. In the discussion on the teaching of number, we shall have occasion to point out groupings of units that make for more efficient mastery than others. These tests of the relative worth of various illustrative materials are still in the embryonic stage, but teachers and supervisors must keep abreast of the progress that is made in this branch of experimental education, and apply the scientific conclusions.

Dangers of Overemphasizing Objective Teaching.—It must be recalled that only new concepts must be developed concretely. Part of the disciplinary aim of arithmetic is to develop in the student that mental power which will enable him to rise above the concrete. We must not only discontinue objective teaching when concepts are no longer new, but we must also be on our guard not to plan too elaborate use of objective material. Recent writers complain that in current teaching there is no unity in the use of objective material, for beans, sticks, papers and marbles are employed in the same lesson. Suppose in the lesson on the two combinations, 7+4 and 7+5, the children add a variety of objects to discover 7+4, 7+5, 4+7, 5+7, 5+ ?=12, ?+4=11, etc.; what harm is done? The purpose is obviously to reduce these combinations to habit; the illustrative material is only an introductory step, a means to an ultimate end. Had the teacher decided to have the children play store

and add miniature bags of flour and boxes of oatmeal, would the resulting "unity in materials" have reduced these combinations to habit more speedily and more easily? Objective teaching usually runs to extremes—it is either neglected or made a fetish.

CHAPTER IV

GENERAL PRINCIPLES GOVERNING THE TEACHING OF ARITHMETIC (CONCLUDED)

GRADATION IN ARITHMETIC

Careful Gradation a Determining Factor in Successful Teaching.—A cause of much of the difficulty which children experience in learning arithmetic is ungraded and poorly graded subject matter. Textbook writers and teachers often take the logical attitude toward the subject. The new topic to be taught is divided into its subtopics and these are presented in logical sequence. But it must be remembered that subtraction changes with each new example, for new factors enter, new combinations are required, and known figures take new relative positions. A child who can subtract 2786 from 8234 may not be able to take 245 from 708, and a child who can solve the second example may find himself puzzled when two ciphers are introduced into the minuend, thus: from 8007 take 3543. Similarly, a pupil may be able to cancel the proper factors in the example $\frac{2}{3}\times\frac{3}{5}$ but may be helpless when the fractions are changed to read $\frac{2}{3}\times\frac{5}{7}$. In the first case, the child divides the proper numerator and denominator by 3 which is seen, but in the second example the divisor 3 is not the same as both the numerator and the dominator to be canceled. Similarly a child who can solve $\frac{3}{5}$ of $\frac{5}{7}$ may not know how to proceed in the example of $\frac{3}{4}$ of $\frac{6}{7}$, for here the common factor, 2, is not seen but must be discovered.

The teacher who has an intellectual sympathy for the

child can take the attitude of the immature pupil and discover, one by one, the successive difficulties that present themselves in the unfolding of the subject. Such a teacher introduces into a lesson a gradation that goes far to remove obstacles from the learner's path. Most children in a well-organized class can readily master the prescriptions of the average course of study. The most usual cause of failure lies in the fact that because of imperfect gradation more than one difficulty is presented at a time. Let us assume that division by a decimal has been explained carefully by the teacher and understood clearly by the class. It does not follow that because the children can solve examples used by the teacher in the course of instruction in which the divisor is .6 or .8 or .15, that they can therefore solve any case that comes under the caption "Division by Decimals." When the example takes the form of $.15\overline{)2.455}$ or $.12\overline{)36.48}$ no difficulty is experienced, but when a slight change is made so that the example becomes $.15\overline{)245.5}$ or $.15\overline{)2455.}$, many in the class are puzzled, for there is an insufficiency of decimal places in the dividend. The writer has seen children able to solve $.15\overline{)65.75}$ but unable to obtain the correct answer to $.15\overline{)657.5}$; able to solve $.15\overline{)657.5}$ but unable to manipulate the necessary elements in $.15\overline{)6575}$; when questioned as to the cause of the difficulty the children replied, "There is no decimal point in the dividend." The reader might assume that the instruction was inefficient and that the children were taught that to solve .15)65.75, "we move the decimal point two places in the divisor and two places in the dividend." Such an assumption would be unwarranted, for when the pupils were asked to explain the division of 65.75 by .15 they said: "We multiply the divisor by 100 and then do the same to the dividend." When they were requested to apply the law to the division of 65 by .15 they moved the decimal point in .15 two places to the right, but since there was no

decimal point in the dividend, they multiplied it by 100 by adding two ciphers and proceeded as follows:

$$\begin{array}{r} 4\ 33\ \tfrac{5}{15} \quad 433\tfrac{1}{3}. \\ 15.\overline{)65\ 00} \\ 60 \\ \hline 50 \\ 45 \\ \hline 50 \\ 45 \\ \hline 5 \end{array}$$

But without these questions they seemed unable to proceed. In the teaching of division by decimals careful gradation demands the introduction of the following steps, each taught only when the preceding one is mastered.

Step	*Reason*
1. 5)54.25	1. This is a known procedure.
2. .5)54.25	2. This introduces the problem of division by a decimal and shows the difference between the known form and the new.
3. Divide by .6, .7, .9, .6)37.86 .7)54.67	3. Drill in dividing by decimals of one place.
4. Divide by .12, .14, .15, .12)38.64 .14).322	4. Applies the same law to decimals of two places.
5. Divide by 1.2, 1.4, 1.14 1.2)38.64, 1.4).322, 11.4)7.98.	5. Introduces the idea that the whole number in the divisor has no influence on the mode of procedure.
6. Divide 32.2 by .14	6. Places in the dividend insufficient and must be made up by the addition of ciphers.

7. Divide 322 by .14.	7. Dividend a whole number.
8. (*a*) Increase the decimal places to 3 places.	8. (*a*) Carrying out same law with examples a little more difficult.
(*b*) Divisor 3 places and a mixed number, e.g. 1.215.	(*b*) Showing that whole number, again, has no influence on mode of procedure.
(*c*) Divisor 3 decimal places, dividend less than 3 decimal places.	(*c*) See Steps 6 and 7.
9. To divide by 240, 2400, 800, etc.	9. While this is not dividing by a decimal, the same general law operates and helps to simplify the process, e.g. $24.00_{\times})87.65_{\times}\overline{42}$
(*a*) 2400)5468.76	(*a*), (*b*) and (*c*) The gradation here indicated is explained above when similar gradation is shown in the division by a decimal, Steps 6 and 7.
(*b*) 200)5.468	
(*c*) 200).5468	

In planning the application of a new principle in arithmetic, the teacher must list all useful forms or types and then so grade these that as the round of applications is met only one new difficulty presents itself to the child.

"LABELING" AND "CRUTCHES" IN ARITHMETICAL ALGORISMS

Excessive Labeling.—In most classrooms it is common practice to require children to label all numbers in the course of solving a problem. Since every quantity that represents something concrete must be tagged, considerable writing is done before the answer is obtained. Many teachers oppose "full forms," which means excessive writing; various abbreviations are taught, but every quantity is labeled; *h. m.* stands for "how many," *l* for "loss," *g* for "gain," *m. p.* for "marked price" or "catalogue price," etc. This labeling is justified by a variety of reasons: It encourages sustained thinking, teaches systematic arrange-

ment of work, simplifies inspection and examination by the teacher, and encourages accuracy.

Upon closer analysis, we may doubt the effectiveness of the habit of " labeling all numbers." It surely is not the method pursued in real life; one thinks about the problem and decides that if a were added to c and the result divided by b, the number of " bales required " will be found. Only the answer is labeled. The more interested one becomes in the problem, the more impatient does he become to reach the end of the manipulations and obtain the answer. It is not uncommon to find children who have obtained the correct answer going over the problem and inserting names to accompany the numbers. Nothing is gained, in the course of solving an example, by forcing the child to arrest the solution in order to associate every number with the concrete object that it represents. Labeling all numbers in solving problems is often an obstructive practice.

" Crutches " or Accessory Algorisms.—The term " algorism " refers to arrangement and form of arithmetical solution. It is common practice to find auxiliary figures and marking taught children when new topics or operations are introduced. To teach a child, in column addition, " to put down the units under units, write tens in the corner, and then carry the figure to the other tens"; in subtraction, to change the figures in the minuend; in division, to make checks under each figure of the dividend as it is brought down—these are illustrations of accessory algorisms or " crutches." Such work takes on the following appearance:

346		7 10 | 14		345
218		3 4		14)4836
347		146		42 x x
236	3	238		63
195	2			56
42				76
				70
				6

The tendency among recent writers is to reject all these crutches for reasons that are serious. (1) It is argued that children who learn to use these aids develop skill in a form that is not the form that will finally be used. (2) Waste of time and energy is caused by teaching one form until a habit is developed and then requiring children to break it so that another habit may replace it. (3) Every auxiliary check tends to keep the process, already understood, a rational one, so that speed is reduced. Crutches may therefore be devices whose ultimate cost is far outweighed by the temporary aid that is rendered. For this reason, there is a growing feeling that all crutches should be eliminated.

Theoretical analysis may bear out the final conclusion, but practical experience often refutes such a sweeping assertion. Teachers who grade their work carefully and give sufficient drill on each step find that many children can be brought to a desired level of proficiency without the use of accessory algorisms. But there are young pupils who for one reason or another cannot master a new solution without the aid of crutches. To force such children to abstain from their use reduces their speed and their accuracy and makes the learning process unduly difficult. It is obvious that children must be encouraged to drop these aids as soon as possible, but not all children can be treated in the same way.

The Correctness of All Algorisms.—Such statements and algorisms as are used must always be mathematically correct. Arithmetic must be made the instrument of developing precision of speech as well as accuracy of thought in dealing with quantities. Textbook writers object to the formula 5 ft. $\times$4 ft. = 20 sq. ft. on the ground that the multiplier cannot be concrete, and even if 5 ft. could be multiplied by 4 ft., the conversion into square feet would be impossible. They suggest the form $5 \times 4 \times 1$ sq. ft. = 20 sq. ft., an algorism which the child hardly understands and which

makes the multiplier no less concrete to him. In solving problems of the indirect type, "A peddler who always sells to gain $\frac{2}{5}$ of the cost, made a profit of $6.20 in a day. What did he invest in his stock?" loose statements like the following must not be tolerated:

$\frac{2}{5}$ = gain
$6.20 = gain
$\therefore \frac{2}{5}$ = $6.20
$\frac{1}{5}$ = $\frac{1}{2}$ of $6.20 = $3.10
$\frac{5}{5}$ = $3.10 × 5 = $15.50

It is manifestly absurd to say that $\frac{2}{5}$ and $6.20 are equal. We must insist on an accurate statement of facts as shown below:

$\frac{2}{5}$ of cost = $6.20
$\frac{5}{5}$ of cost = ?
$\frac{1}{5}$ of cost = $\frac{1}{2}$ of $6.20 = $3.10
$\frac{5}{5}$ of cost = $3.10 × 5 = $15.50 Ans.

Children can be shown the importance of accurate statements by referring to one of the traditional fallacies discussed in logic. On the supposition that a barrel half full contains as much as a barrel half empty, the following may be evolved:

$\frac{1}{2}$ bbl. full = $\frac{1}{2}$ bbl. empty
Multiply both sides of equation by same number, 2
$\frac{2}{2}$ bbl. full = $\frac{2}{2}$ bbl. empty
bbl. full = bbl. empty

No aim in arithmetic can be attained unless it is made a goal in all phases of the work. To insist on accuracy in number manipulations and to neglect accuracy in statement and arithmetical formulation lead ultimately to inaccurate and loose thinking in all arithmetic.

INCULCATING HABITS IN ARITHMETIC—THE DRILL

Arithmetic a Subject of Habits.—School studies may be grouped into content subjects and formal subjects. History, descriptive geography, literature, nature study are illustrations of the first, while arithmetic, grammar, locational geography and map study come under the second. The former group of studies is rich in theory, story, and human relations; the latter sums up settled facts and fixed associations. The controlling purpose of all formal subjects is to reduce necessary associations to habit. In arithmetic, the combinations of addition or multiplication, the mode of dividing, and the procedure in cancellation are only a few of a host of habits that must be inculcated so that correctness of result with a minimum of conscious direction is assured. A subject in which so many habits must be developed is, in the very nature of the case, one in which the drill lesson plays no small part.

What Habits Should Arithmetic Develop?—The manifold habits that can be developed through the mastery of so formal a subject as arithmetic can be grouped under three heads. There are first the *habits* of *subject matter*. Every item in the multiplication tables, in the forty-five additive combinations, the reverse of these, the facts in the necessary tables of weights, all the modes of computation, notation of decimals, short cuts, per cent equivalents of common fractions, modes of checking answers—all these constitute the subject matter which must be so thoroughly mastered that they in turn master the child. An enumeration of all the subject matter habits that must be inculcated will lead only to a repetition of the content of the minimum course of study (page 22).

The second set of habits are the *habits of procedure*. Throughout the teaching of arithmetic, teachers must insist on a systematic arrangement of all written solutions; on accepted forms of ruling of bills, cash account and busi-

ness forms; on scrutinizing an answer and submitting it to a checking device before going to the next problem; on examining a problem before solving it and writing out what is given and what is to be found; on labeling necessary items in the solution of an example; on reducing the numbers in problems that seem difficult to much smaller ones and then attempting to evolve a solution; on attempting to translate new problems into diagrammatic form and then seeking a solution; on neatness and legibility of all figures and words. These procedures are constantly emphasized by the teacher until by dint of repetition they sink deep and are performed through force of habit.

Mental habits constitute the third group. Throughout the teaching of arithmetic we strive to develop a sense of accuracy, increased concentration, power to imagine new relations, power to reason and judge. It is hoped that as a result of years of study of arithmetic children will acquire a mode of attacking a problem and power to deal with quantitative relations.

These three sets of habits mark the permanent results of the subject, the final ends which the teacher seeks to attain. Scientific standards in arithmetic for determining progress and capabilities of pupils measure the habits which have been inculcated in children.

Objections of Functionalists to Formal Drill.—In recent pedagogical literature strong objections are often voiced against formal drills on the ground that mere mastery of abstract combinations in arithmetic is no guaranty of equal proficiency in these very combinations as they occur in practical solutions. The child may be able to give the correct answer to $5+7=?$, $5\times7=?$, $5\times?=35$, $?+7=12$, and, nevertheless, be unable to solve simple problems of the type of, "Had 12 cents and lost 7; how many cents have I left?" "If each of 7 boys has 5 marbles, how many have they all?" "Children need 35 marbles for a game in which 5

are to play; how many marbles must each bring?" Although the child knows the combinations as such, he may not be able to apply them when they are made to function in a practical problem. It is argued, therefore, that as the child solves numerous problems in which these combinations occur, he acquires not only a perfect mastery of the latter, but also becomes expert in their application.

This objection to formal drill will be regarded by the practical teacher as an extreme viewpoint borne out by theoretical conjecture rather than actual experience. While it is true that mastery of an abstract combination of numbers is no guaranty of ability at application, it does not follow that there must be no formal drill in the tables. A mastery of the items in the tables is a step towards developing power to apply these in practical problems. Practice in certain seemingly isolated abilities is necessary in many arts. The singer practices scales, intervals, and range; the basketball player practices tackling, blocking, shooting the ball; the handball enthusiast practices forms of serving; in all these cases drill in isolated abilities which are the constituent elements of the larger activity is found necessary. True, ability to sing certain scales is no guaranty of ability to sing effectively the song in which they occur; nor does expertness in tackling or in serving inevitably lead to expertness in playing the game or basket ball or handball. In all arts, as in the art of arithmetic, there must be two forms of drill—formal drills in isolated abilities and functional drills, practice in the application of these abilities. Many writers use the term "drill" only for the formal drill; and the expression "modes of application" for all functional drills or drills in problem form.

Forms of Drill.—It is evident from the preceding discussion that associations that occur and recur in the same form must be subjected to drill. In arithmetic these drills must take a variety of forms. They must be not only oral,

but written as well. Not only must children be practiced in 5×8, 6×8, 7×8, 8×6, 8×7, in oral work, but in the written period they must be required to solve 268×8, 675×8, 587×8, etc., so that ability developed in the one form may be transferred to the other. Nor must it be assumed from these illustrations that all drills must take an abstract form. Successful work demands equally extended drills in problem forms as well as in abstract forms. Children learning the first case of percentage should, therefore, be subjected to both of the following forms of drills:

Drill in Abstract Form:

Find 5% of $500, of $5000, $50, $5.

Find 12% of $800, $80, $8000, $8.

Find 1% of 525, of 52.50, of 5250.

Find 10%, 15%, 24%, 1%, etc., of $900, etc.

Drill in Concrete or Problem Form:

Goods costing $52.50 and sold at a profit of 10% bring what gain?

A man who owes me $7200 can pay only 30% of his debt; what do I receive?

2300 cubic yards of dirt are to be removed. How much has been removed when the contractor completes 40% of his task? etc.

While drills in arithmetic may be in oral, written, abstract or problem form, the variety of ways in which these four forms can be given is almost beyond count. If the reader will turn to the discussion of Variety of Objective Material (page 71), and to the variety of drills offered at the end of the lessons on Teaching Numbers (page 149), Teaching a Combination (page 160), Teaching Addition (page 150), Teaching Multiplication (page 184), Counting (page 162), he will readily realize how rich the subject of arithmetic is in its possibilities for variety of drill. To tabulate this variety here would entail needless repetitions and would wrench subject matter out of its natural sequence and association.

Characteristics of an Effective Drill Lesson in Arithmetic.—The discussion of the drill lesson in arithmetic cannot be concluded without a consideration of the qualities of effective drills. Since the final purpose of a drill lesson is to insure the functioning of knowledge as habit, the laws governing the drill lesson must be identical with those controlling habit formation. From this viewpoint, the following comparative table may make clear the pedagogical requirements of a drill lesson in any subject.

Habit Formation	*Effective Drill Lesson*
1. *Purpose.*—The final purpose in all habit formation is to secure automatic reaction so that maximum speed and highest degree of accuracy are attained. What is done from force of habit is done more quickly and with less chance for error than that which is performed through conscious direction. The assumption, however, is that the situation presents no new elements and that the usual reaction is required. The child who spells the word "occur" because the writing of it is a habit does so in less time and with less chance of making an error than his classmates who decide on each successive letter. But should the word be changed to *occurred*, a new	1. *Purpose.*—In all drill lessons the aim is to develop such a perfect mastery of the associations involved that the presentation of one element will instantaneously and automatically recall the others. When this control is developed the recall occurs more speedily and with a degree of accuracy unknown in a reaction in which conscious effort and reason prevail. The child who has been successfully drilled in adding orally 27 and 56 says, "27, 77, 83." The child who must think of the steps says, "27 and 50 will give 77; 77 and 6 will give 83." Not only is more time required by the second pupil, but actual experience demonstrates immediately that such a child is more likely to err.

situation arises and the law of habit no longer holds.

2. *Steps.*

(*a*) *Motivation.*—Without the child's coöperation habits cannot be formed effectively. It is therefore wise to lead children to a thorough understanding of the values of the habits that they are to develop. As far as possible every habit to be inculcated should be motivated, should be interpreted in terms of the child's needs.

(*b*) *Comprehension.* — Before any habit can be formed there must be a thorough understanding of the reaction that is to be habituated. Children must understand what constitutes politeness or impoliteness, neatness, accuracy, or correct standing posture, before they can develop desirable habits in these directions.

(*c*) *Repetition.* — With motivation and comprehension assured, we may proceed to the repetitions

2. *Steps.*

(*a*) *Motivation.*—Much of the tedium of the drill lesson can be removed if the pupil regards the drill as necessary. Before beginning the round of repetitions, the drill lesson must be motivated. This can be done in any one of the number of ways that are explained and illustrated in the discussion of the topic, Motivation. By a few judicious questions and suggestions, the teacher can lead children to realize that ability to derive 5×7, 6×7, and 7×7, or the items in the linear measure, will not suffice; that repetitions are necessary if these facts are to be reduced to second nature.

(*b*) *Comprehension.*—No drill lesson must be attempted unless there has been a period of explanation. No combinations, processes, modes of manipulations, or number facts must be subjected to a drill process without reasonable certainty that the children understand the ideas and the relationships involved.

(*c*) *Repetition.*—The subject matter that is understood and motivated may now be subjected to the round of repetitions that constitute the essence

that are to form the habit grooves in the nervous system. It must, however, be borne in mind that mere repetition will not suffice; these repetitions must be *regular*, *frequent*, for *short periods*, and *without lapses*.

of a drill lesson. As in habit, so in drill lessons, these repetitions must be *regular* and *frequent*. Repetitions that come once a week or a fortnight, or that occur often on one day and are neglected for many days after that, will not reduce knowledge to habit. Drills in arithmetic should come daily in special periods regularly set aside for them. Care must be taken, however, to keep the drill periods *short*. More effective work can be done in two drill periods of five minutes each than in one period of twelve or fifteen minutes. This fact has been demonstrated experimentally, but it has not yet been decided scientifically how long to make the drill periods for different grades and different operations in arithmetic.

(*d*) *Attention to the Repetitions.* — In acquiring a habit it is necessary to attend to each repetition so that the coördinations are consciously made. As soon as the pupil masters the reaction, habit sets in and thereafter masters the pupil.

(*d*) *Attention to the Repetitions.*— Mere drill lessons without attention in the focus of consciousness reduce themselves to aimless repetitions that do not lead to mastery of set subject matter. Children who repeat "$5\frac{1}{2}$ yards equal 1 rod" and give themselves to a stream of pleasant imagery of games and associated experiences are merely mouthing words and producing sound. The equivalents, $5\frac{1}{2}$ yards and 1 rod, are not being reduced to habit. Children who write tables as punishment go through mechanical penmanship without consciously attending to the associations involved. Such drills deceive both children and teachers and involve no little waste in time and energy.

(e) *Further Steps in Habit Formation.*—The analogy between habit formation and the development of effective drill ends here. Such additional suggestions as are given in habit formation, e.g., a vigorous initiative, reminders in the form of mottoes, badges, etc., are of importance in habit inculcation but have little application in drill lessons.

(e) *Introduce Conditions that Offset Monotony Inherent in Drill Lessons.*—A few mechanical conditions must be borne in mind if the drill lesson is to be freed from much of its inherent monotony. Chief among these we must mention the following:

(i) SPEED.—The more rapidly the lesson is conducted and children called on, the more effective is the drill.

(ii) AVOID REGULARITY OF SEQUENCE.—In calling on pupils, teachers must not follow the sequence suggested by the alphabet or by the arrangement of seats. Calling children in regular sequence makes for attention only among those who are about to recite. Such a degree of comfort experienced by the class during a drill is an infallible sign of weakness in the conduct of the lesson.

(iii) CHANGE THE FORM.—Variety of statement should be introduced as often as possible. The statement 5×9 may be changed to 9×5; $?\times9=45$; $5\times?=45$; 5, 45 and ?; 9, ? 45. The form "4 quarts = 1 gallon" may be given as 1 gallon = 4 quarts, 2 quarts = $\frac{1}{2}$ gallon, 8 quarts = 2 gallons, 6 quarts = $1\frac{1}{2}$ gallons, etc. As the statement is varied attention is reinforced and directed to the combination to be mastered.

(iv) APPEAL TO MANY SENSES.—It is not wise to confine drills to oral forms only. Effective drills are carried on by appeal to visual and muscular senses as well. Children are told to look at

$7 \times 9 = 63$, $9 \times 7 = 63$, which are written on the blackboard. They are then told to shut their eyes and "see" what was before them. This visualization makes a deep impress on many minds. Other children find that number facts that are written and rewritten, whether in the air with the finger or on paper with pencil, will make for thorough mastery.

(v) CAUTION WITH CONCERT WORK.—Concert work used infrequently and for very short periods may prove effective. But as a regular mode of drill it has limitations which make its use doubtful, for it discourages attention, makes limited sense appeals, and introduces a soporific rhythm.

ACCURACY AND SPEED IN ARITHMETICAL OPERATIONS

Relation of Accuracy to Speed.—Because of the exactness of mathematical relations, the standard of accuracy in arithmetical calculations is necessarily one hundred per cent. Of two children equally accurate, other conditions being equal, the more rapid worker has the advantage in all practical situations. It is desirable, therefore, to develop as high a rate of speed in calculation as is thoroughly consistent with absolute accuracy. Speed acquired at the cost of accuracy leads to retardation rather than acceleration of work.

It has been a traditional belief among teachers that as accuracy increases speed develops in approximately equal proportions. Experimental evidence shows conclusively that these abilities do not develop together, for schools that rank first in accuracy are often much lower in the scale of

speed. Both accuracy and speed are qualities of arithmetical work which respond readily to systematic and regular drills and show steady increase when appropriate exercises are devised.

How Can Accuracy Be Increased?—Accuracy can be developed by continually emphasizing a few simple practices throughout the teaching of arithmetic. Irregular attention to these devices is productive of no results; regular insistence on these will gradually bear fruit. These suggestions for developing arithmetical accuracy must now be summarized.

1. *Develop Habit in Fundamental Operations and Combinations.*—Our analysis of habit showed that habituated reactions are more accurate than those that result from deliberation. In all drills on necessary operations and combinations in arithmetic, the ever-guiding aim must be the development of a set of habits which will guarantee results of absolute accuracy.

2. *Legibility of Figures.*—Much inaccuracy arises from the fact that children's figures are poorly formed or so carelessly or irregularly placed that correctness of final results is impossible. Civil service boards throughout the country are now testing for this quality of penmanship in candidates for clerical positions. Legibility, when applied to figures or to penmanship, sums up a complex of qualities. It refers to "ability to read" under every variety of condition. In multiplying by 204 we often find children so careless in the placing of the various figures of partial products that correctness is rendered impossible.

$$\begin{array}{r} 736 \\ 204 \\ \hline 2944 \\ 147\ 2 \\ \hline 149964 \end{array}$$

In the accompanying illustration, the 2 of the second partial product is erroneously added to the 4 of the first partial product; 9 is brought down and 2 and 7 give the second 9; the 14 completes the incorrect answer. Errors in every variety of number manipulation can be traced to similar carelessness of arrangement as well as to malformation of figures.

The conclusion for the teacher in all these observations is simple. Penmanship drills must include the writing of figures as well as letters. But just as penmanship exercises give practice in writing words as well as letters, so, too, figures must be written in a variety of combinations and groupings so that the child develops ability to write figures legibly and to place them accurately. Scales for measuring penmanship should include specimens of figures in a progressive order of merit.

3. *Ability to Copy Figures.*—Closely related to legibility of figures is the ability to copy figures accurately. Many teachers regard errors in copying figures as one of the chief sources of inaccuracy in arithmetic. These errors are usually one of three forms, viz.:

(*a*) introducing new figures, i.e., copying 37 for 7 or 57;

(*b*) interchanging figures, i.e., copying 37 for 73;

(*c*) omitting figures, i.e., copying 37 for 307.

Brown and Coffman[1] tell us that experimentally they have ascertained that the first type of error is more common than the second in the ratio of 28 to 11; that figures are omitted one-third as frequently as they are miscopied; that pupils below the average in speed miscopy more frequently than those above the average; that the lower the grade, the greater the number of errors due to miscopying. In addition to practicing children in writing figures, regular exercises should be given in copying figures in a variety of groupings. This can be made a form of busy work or homework,

[1] Brown and Coffman, "How to Teach Arithmetic," pages 54–55.

in which children check one another's results. There is little that the teacher can do to reduce errors in copying, aside from dignifying these exercises, giving them frequently, and ascertaining whether pupils who do not improve are suffering from some form of eye ailment. The greater the opportunities for practice, the surer are normal children to develop ability to copy figures accurately.

4. *Critical Examination of All Results.*—The habit of testing the result of a computation, so thoroughly ingrained in all accountants, must be developed in children. They must be taught how to approximate results and how to apply various checks. The various modes of testing answers and the system of checks applicable to each of the fundamental operations will be given in detail in Chapter IX.

5. *Habit of Planning Work and Seeking Correct Interpretation of Problems.*—A cursory examination of children's results in arithmetic shows that the untrained mind usually attacks a problem in almost feverish haste without attempting to familiarize itself with the facts and conditions of the problem or to understand clearly what is demanded. The general appearance of the problem recalls a seemingly similar type and the solution is often half completed before the child realizes that new conditions have been introduced. Frequently the confusion that results is too great to permit clear thought and successful solution. Children must be taught how to analyze problems and plan for their solution. The specific method and devices for teaching children modes of attacking problems must be postponed to the chapter on Problems.

6. *Accuracy of Statement.*—We must reëmphasize the fact that accuracy of statement makes for accuracy of thought and that looseness of statement encourages looseness of thought. The suggestions given earlier in this chapter for accuracy of algorism and precision in the use of

language in arithmetic must now be recalled as illustrations of this final suggestion.

How Can Speed Be Increased?—1. *Develop Accuracy.*—While it is true that the accurate worker in arithmetic is not always speedy in his manipulations, it is, nevertheless, axiomatic that inaccuracy is the greatest handicap in striving to attain speed. The plea made for one of our school systems, that while it ranked below average in accuracy, it nevertheless, stood much above average in speed, did not impress one of the school superintendents who saw in the explanation only the fact that it " takes us less time to get a thing wrong than it does in the average school system." The habit of accuracy is the greatest single factor that makes for speed.

2. *Acquisition of Necessary Habits in Arithmetic.*—Reactions that are automatic are always more rapid than those that occur through direction by higher centers. The more effort spent on inculcating in children the complete round of necessary subject habits, mechanical habits, and mental habits, the more is the teacher contributing towards increased speed in the arithmetical manipulations of the pupils.

3. *Limit Time for Assigned Work.*—Experience shows conclusively that most rapid workers in arithmetic have developed speed as need for it was felt. The inexperienced bill clerk or bookkeeper finds that the conditions of business demand more rapid computation. He puts an extra strain on consciousness, gives himself more completely to the task at hand, and finds that he can increase the speed with which he works. It is wise, therefore, to limit the time for all work in arithmetic. After the assignment is made the children should be told that they will be held responsible for the entire task at the end of a set period. The class feels the strain and sets to work with greater concentration and more intense application. To allow children to take the

time that they deem necessary discourages concentration and dissipates effort.

4. *The Use of Short Cuts.*—Much time can be saved by teaching children to use short cuts when longer processes are understood. When short cuts are used not at the teacher's direction but from force of habit, economy of time and effort is achieved.

5. *Reduce Formal Analysis.*—A cause of much of the slow, cumbersome solution of problems can be traced to excessive analyses and algorisms with their many "labels." We must look upon formal analysis as an aid to systematic thinking when a type of problem is new. But just as soon as children give evidence of a thorough comprehension of the thought involved, formal analysis must be abandoned. It must be a constant aim to reduce the language accompanying the solution of a problem to an absolute minimum.

6. *Abandon Crutches.*—Auxiliary figures, when justified at all, may be used in the introductory stages of a new process; but when mastery in the solution taught is assured, the crutches must be eliminated. All reminders in subtraction, in division, or in fractional solutions must be dropped as soon as the progress of the children will permit.

7. *Use a Minimum of Figures.*—Teachers must also strive to have their children use as few figures in the solution of a problem as possible. The problem given to a sixth year class was: "The census shows that 55 per cent of the population of a city of 145000 inhabitants is colored; 20 per cent of the negro population is in skilled trades; how many colored people have skilled trades?" The model solution that the teacher required was as follows:

145000 inhabitants
.55
725
725
79750.00 colored population.

$\frac{1}{5}$ of 15950 people = 15950 colored people in trades. Answer.

Judging by the number of children who obtained the correct answer, there was no need of insisting on so many labeled numbers and so careful a statement of each operation. A more economical solution should have been given to the class. The following forms are submitted for comparison:

145000
.55
725
725
5)79750.00
15950 colored people in trades. Ans.

$\frac{1}{5}$ of $\frac{\overset{11}{\cancel{55}}}{\cancel{100}}$ of $145\cancel{000} = 15950$, colored people in trades.

The time saved in solving this one example by the shorter method is not of serious consequence, but it saves the child from wrong habits of work which require needless penmanship and useless repetition of numbers.

8. *Assign Work to Children According to Their Ability.*— The practice of assigning a fixed number of examples to all children in a class is responsible for the development of habits of slow work among abler pupils. When uniform assignments are made regardless of ability, the more capable complete their work in relatively short time. Some of these pupils continue working rapidly for the satisfaction they feel in "finishing first," while others soon learn that even though they work slowly they nevertheless complete the given task in good time. Gradually they acquire habits of divided attention which cling with a persistence that renders futile later endeavors at concentration.

THE PLACE OF DEFINITIONS IN ARITHMETIC

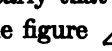
The Exaggerated Importance of Formal Definitions.— Arithmetic has a terminology of its own. For effective instruction and for application of arithmetic to real situa-

tions of life, it is necessary that children know the language of arithmetic. But it must be remembered that ability to define the important terms is no guaranty of their comprehension. The child who glibly recites a formal definition of *number*, *unit*, or *ratio* is repeating a statement imposed by his teacher and formulated by a person of mature mind and thorough insight into the subject. Nor are all definitions correct and immutable; most of them attempt to summarize prevailing characteristics. The definition of a fraction, which is most frequently given by textbooks is the traditional " one or more of the equal parts of *a* unit." But $\frac{3}{4}$ is not always $\frac{3}{4}$ of 1, i.e., 3 of the 4 equal parts into which *one* unit has been divided; $\frac{3}{4}$ may refer to $\frac{1}{4}$ of 3. The original definition of a fraction does not suggest, even remotely, this second interpretation of a fraction. Formal definitions are not the necessary aids which older pedagogy thought them.

Necessary and Unnecessary Definitions.—But it must not be assumed that all definitions may be eliminated in the teaching of arithmetic. We must draw a distinction between serviceable and useless definitions. It is obvious that the primary grades must be freed from formal definitions. In these classes the children are too young to formulate the definitions themselves and too immature to read meaning into those that are given to them.

Definitions that Describe v. Definitions that Classify.—It may be helpful to separate definitions into two classes depending upon their functions. Some definitions are purely descriptive in nature and sum up those general qualities which label a whole class. The definitions of addend, factor, multiplicand, multiplier, dividend, divisor, subtrahend, denominator, are illustrations. Nothing is gained by having formal statements memorized for definitions of these terms. If children can recognize these elements in arithmetic and can give examples of any one of them, they

have sufficient knowledge to enable them to pursue their work intelligently.

But there is a second group of definitions which serves as a basis for classification. The definition of rectangle, triangle, parallelogram, common v. decimal fraction, proper v. improper fraction, does more than state general characteristics. The child who does not know how to formulate a definition of a rectangle may not see clearly that a square and an oblong are rectangles but that the figure  is not in the same class. Children who have vague notions of a rectangle but who can find the area of any rectangle often solve a problem which calls for the area of a parallelogram whose base is x feet and whose inclined side is y feet, by multiplying x and y. We may conclude, therefore, that in arithmetic only those definitions that serve as a basis for useful classification should be taught in the upper grades, while those that are purely descriptive may be neglected and emphasis placed on recognition, comprehension, and application of the terms.

Suggestions for Teaching Definitions.—Where definitions must be developed it will be found helpful to incorporate the following suggestions dictated by experience:

(1) The *definition* should come *after* the lesson has been developed and the *functions* and *characteristics* of the term in question have been *thoroughly grasped* by the child. The lessons on Meaning of Fraction, and Multiples, will give ample illustrations of this suggestion.

(2) The *definition* must be *formulated by the class*, various children making improving contributions to the final statement. However inadequate the language used by the children may be, *they* must, nevertheless, *suggest* the *thought elements* in the definition, for only as they succeed in doing this are they giving conclusive evidence of comprehension.

(3) After the component ideas have been contributed by the pupils, the *teacher* may *modify* the *language*.

(4) The final statement need not be memorized by the children. It will suffice to have them *learn the component ideas* in a definition. When called upon to give a definition of a rectangle, children should think of "four sides" and "four right angles" and should then attempt to express these two ideas in the best sentence they can formulate. A stereotyped definition is rarely necessary.

(5) It will be found a helpful procedure to train children to precede each definition with an original example. The child called upon to give the definition of *factor* or *rectangle*, says for the first, "Five is a factor of ten or fifteen—factor is a number that divides another number evenly"; and for the second, "The window pane is a rectangle—a rectangle is a figure having four sides and four right angles." Children are often able to give correct definitions but lack understanding and therefore ability to apply their knowledge. The original illustrations which precede the definition give unmistakable evidence of comprehension.

RATIONALIZATION OF ARITHMETICAL PROCESSES

Importance of the Problem.—An ever-present problem in the teaching of arithmetic is the determination of the extent to which number facts and processes should be based on reason or on authority. Much time is spent explaining to children what, in their immaturity, they cannot understand. Excessive rationalization is an expression of an adult interest in the subject rather than an answer to the child's curiosity. Such teaching involves waste of time and mental energy. To eliminate all attempt to rationalize any part of arithmetic results in its thorough mechanization. Such instruction is deadening and robs the subject of every vestige of disciplinary worth. A plan of instruction in arithmetic which will set intelligent limits to the problem of rationalization results in economy of time and energy and

retains the mental vigor which should accompany the study of arithmetic.

Older modes of instruction in arithmetic were characterized by severe memory drills and constant imitation by the pupils. Facts were accepted on the teacher's authority and were at once applied to a variety of situations and problems regardless of their practicability. Thorough habituation was the aim; blind rote work seemed to be the only means of attaining it; the teacher demonstrated and the child imitated. The reader may have a vivid recollection of memorizing the law, "To divide a fraction by a fraction, invert the divisor and proceed as in multiplication." When this was mastered, it was explained and applied to a few examples by the teacher. The class was then set to work to imitate the mode of solution by finding answers to other examples.

The reaction set in with the popularization of pedagogy. Deductive teaching was condemned; inductive teaching, the development method and objective demonstrations were extolled. Every step was explained, every process rationalized, every relationship discovered, every number properly labeled. Children were so busied understanding the "why" that they lost the "how" and the "what." The elaborate devices of bundling sticks into tens and hundreds, of packing beads into bags of tens, hundreds or thousands in preparation for the teaching of notation to seven-year-old children are illustrations of the type of method prompted by the rationalization fetish in arithmetic.

To-day there is a clear realization that there are limits to both rationalized and mechanized teaching which must not be transcended. Progressive tendencies in arithmetic teach the need of differentiating mechanical from rational processes and seeking to reduce the former, without explanation, to the plane of habit while preserving for the latter all rationalizations.

There are authorities on methodology of arithmetic who set themselves against all rationalizations except solution of problems. These writers seek to emphasize arithmetic as an art rather than as a science. Their position appeals to many teachers and must, therefore, be scrutinized. We must summarize their arguments and note their final worth.

Arguments of Opponents of Rationalized Arithmetic	*The Criticism*
1. What is to be used mechanically in everyday life must be taught mechanically in school. All the processes of arithmetic are applied in post school days from force of habit and should therefore be inculcated by a series of repetitions without rationalized explanations.	1. The argument is a *non sequitur*. The aim in all teaching is to mechanize all necessary knowledge so that it is applied from force of habit while the mind is freed for new adjustments. The products of the United States or the facts of the most important tables of weights and measures should be so thoroughly mastered that they are recalled when necessary without appreciable conscious effort. But we do not counsel teaching a list of products without leading children to understand the physical forces that explain a country's products. Nor do we tolerate methods of teaching which impose a table without having children first see the units and discover the various relationships. The present tendency in teaching is to proceed from rationalization to mechanization.
2. Modern psychology sets "habits" as the final goals in all instruction. The immediate aim is therefore to reduce knowledge to habit. To rationalize is to focus con-	2. True, habits make for mental economy, but unless habits are preceded by appeal to reason the individual is helpless when new conditions arise. To habituate without understanding reduces life to stupid repetitions utterly dependent on the avoidance of change.

sciousness on a reaction that should become automatic.

3. Efficiency is not impaired if children do not know the justification for a process they perform automatically; why, therefore, "stir up a hornet's nest by explaining borrowing in subtraction, carrying in addition or the mode of building up tables of multiplication or of weights and measures? "

3. So long as there is no variation from the type, efficiency in a process is not determined by comprehension of it. But new conditions arise constantly in arithmetic, even in the mechanical operations. A child capable of subtracting 276 from 854, when confronted for the first time by the example, 804 − 276, may be unable to find the answer because of the cipher. A pupil who understands the mode of solution applies what he knows and finds the answer by following the same steps. Rationalization gives an intelligent insight into a process and enables the pupil to meet new conditions.

4. Economy of time demands an elimination of all attempts at rationalization in every operation in arithmetic.

4. Immediate economy of time results from the elimination of all explanations and the mechanical imposition of knowledge. But in the long run this economy is dissipated by the helplessness which such methods develop. If all geography and history were taught through repetition without explanation, in order to establish fixed associations, even more time would be saved, but in the long run the loss of power would far outweigh the gain in time.

Principles that Govern Rationalization in Arithmetic.—

1. No process should be rationalized unless it is important. Square root and compound proportion, when prescribed in a course of study, are illustrations of the operations that do not merit rationalization.

2. When an arithmetical situation or type occurs in

variable form, it is necessary to rationalize it. This suggestion applies to problems in the various branches of arithmetic.

3. When there is doubt as to children's comprehension of underlying reasons, it is wise to omit all rationalization. Any attempt to explain to fifth grade children that "increasing the denominator decreases the value of a fraction," "decreasing the denominator increases the value," or the process for finding the greatest common divisor or least common multiple, division by 208, is usually doomed to failure.

4. When repeated use and increasing maturity will reveal the reason for a process that cannot be explained early in the school course without the expenditure of much time and effort, it is advisable to postpone the rationalization. Division of fractions; multiplying a quantity by a fraction decreases the value; cancellation—these are illustrations of arithmetical operations or phenomena which may be presented without rationalization but which acquire meaning through extended application and the more mature reason of later years.

5. The same standard of rationalization must not be applied to all children regardless of native ability. Children who give evidence of power in arithmetic may be led to see reasons in processes that must be taught didactically to others. Children who experience no little difficulty in mastering the essentials of arithmetic may acquire the habit of dividing by a decimal without being able to explain each step; their more gifted classmates may, however, be initiated into the reasons. To some children rationalization is a source of confusion; to others, it serves to add interest and give a deeper grasp of the essentials of the subject.

No hard and fast rules can be laid down to govern all classes, all children, or all topics. The teacher must make

the adjustment which will be most effective in a given situation.[1]

Transition from Rationalization to Habituation.—In concluding the discussion of rationalization in arithmetic we must note the stages by which a process is reduced to habit. These may be summed up under four steps:

1. The teacher demonstrates the process at the board. Questions are asked and the children's coöperation is enlisted in every way as the explanation is evolved. Throughout this rationalization the teacher alone solves the problems used for illustration. The object of this initial step is to insure comprehension by the children.

2. The pupils are now required to apply the new law of arithmetic to problems, all work being done at the blackboard. The teacher scrutinizes the work, points out individual and general errors, and encourages the hesitant pupil. The aim is now to clarify the children's ideas.

3. The progress of the pupil now warrants seat work. Examples are given out and are solved by the pupils at their desks. The teacher maintains supervision which is rather indirect and concerned primarily with the weaker

[1] Compare these principles governing rationalization with those offered in the Report of the U. S. Bureau of Education, No. 13, on Mathematics in the Elementary Schools of the U. S., 1911, page 107.

1. "Processes of frequent occurrence should not be rationalized, e g., correct placing of partial products in multiplying by a number of two or more orders.

2. "If a process does recur in the same manner, but is so little used in after life that any formal method of solution would be forgotten, then the teacher should 'rationalize' it. The process of finding the square root of a number illustrates this fact."

3. "When in doubt as to the frequency of occurrence of a process, it should be taught both mechanically and through rationalization." "The division of a fraction by a fraction is frequently taught both 'mechanically' and 'by thinking it out.'"

4. "When a process occurs in variable form, it must be rationalized. Problems come under this direction."

pupils. The purpose of this third step is to afford drill through variety of application so that the process of habituation is set in active motion.

4. The final stage is characterized by assignments for homework or study periods. The object is now to secure independent application.

It is evident to the practical teacher that these four stages in the transition from rationalization to habituation may be modified, enriched or contracted as changing conditions suggest. Only when they are used with this freedom can they serve any classroom purpose.

TESTING ABILITY IN ARITHMETIC

It is not the purpose of the author to offer an extended treatment of the place of examinations in school instruction, their conduct, their organization, systems of marking, and such other pertinent questions as apply to the teaching of all subjects, but rather to set forth suggestions that have special application to the subject of arithmetic. The general subject of examinations is given with some degree of thoroughness in books on general method.

Objects of Examining in Arithmetic.—There are two sets of reasons which make stated examinations desirable as an integral part of instruction in arithmetic. These reasons are the chief concern of the pupil or the teacher and, therefore, group themselves under two heads.

1. *Objects Concerning the Child.*—From the point of view of the pupil, the examination may be given: (*a*) to lead the child to take stock of his weakness and thus to exert greater effort in special directions; (*b*) to stir pride in the progress indicated by a high rating, and thus feel encouraged to continued endeavor; (*c*) to serve as a motive for reviewing the work of a part of the term.

2. *Objects Concerning the Teacher.*—To the teacher the

examination serves very different purposes. It may be used as an index of the knowledge status of the pupils. It is necessary to ascertain, in a variety of ways, the ability of each pupil and the state of comprehension of old information before new topics are attempted. Properly organized examinations give this information. A second and more vital purpose is to discover the effectiveness of the teacher's own methods of instruction. After a careful examination of the answers the teacher sees at once which points were taught well and which will bear reteaching by new methods. Examinations conducted with these aims in mind may lead to constructive criticism and improved modes of presentation.

Need of Aims in Testing as Well as in Teaching.—We are fully alive to the need of definite aims in teaching, but in giving tests we collect questions and problems on grade work with no governing purpose. A test must be designed to disclose individual weaknesses and the degree of mastery of *specific* branches of arithmetic. A test of ten examples containing one on column addition, one on finding an area, one on division, one on decimal notation, one in addition and subtraction of fractions, one in division by a fraction, one in reduction of denominate numbers, and three in problems involving finding the whole when a fractional part is given leads to a composite result that has little pedagogical significance. Let us assume that a child scores 60 per cent in such a test or that a class averages 55 per cent. What inference can be made that will be useful in the later teaching of that class? Such a composite examination paper may have its place at the end of a term or when a rating or "mark" is required for some reason, but as a test it is devoid of instructional significance and has no educative influence.

In organizing a test, a fixed purpose must be set first, so that the questions bear directly on the aim. An examina-

tion may be given to determine ability in any one of the four fundamental operations, in fractions, in decimals, in copying figures accurately, in use of tables of weights and measures, in accuracy of manipulation, in speed of manipulation, in problems that follow the type, or in thinking problems that vary from the type. Where the teacher is testing for accuracy, no credit is given for wrong answers obtained by model solutions; where the test is for reasoning ability, due credit is given for intelligent mode of solution, even though the result is incorrect; where the test is to ascertain the degree of speed, the time must be limited. Such specialized and differentiated tests reveal the weaknesses of each individual, indicate where further drill is needed and what changes must be made in the course of study. Illustrations of examinations in arithmetic that lead to constructive conclusions and that are scientific in their formulation will be found by referring to the Courtis tests, series A and B, in Chapter XVII and in the Appendix.

EXPERIMENTAL RESEARCH IN ARITHMETIC

Recent experimental studies concerning themselves with the teaching of arithmetic emphasize problems that have hygienic and physiological influences on the pupils. The results obtained in this comparatively new field are meager and crude and do not warrant the formulation of scientific conclusions. In a general way these researches tend to emphasize warnings that must ever be kept in mind. We may summarize these briefly in the concluding paragraphs of this chapter.

Many of these experimental researches point to the dangers of overdrill which produces not added power in application, but a state of mental stupefaction. Teachers who labor on the assumption that no amount of drill is excessive are confronted by evidence which seems to prove the danger of drills that transcend ordinate limits. Drill

which continues beyond the point necessary to develop ability essential in arithmetical operations involves waste of teaching effort and makes for arrested mental development.

Other experimenters are gathering data which may prove that overspeed in arithmetic may produce a mind as enervated as that which results from overdrill.

Important researches deal with the time cost of arithmetic and establish the fact that there is no general arithmetical ability; that a child may be proficient in one branch of arithmetic and yet may be deficient in another; that, therefore, children must not be harassed by arithmetic.

German students of experimental didactics are concentrating on types of mind and the relation of each type to efficiency in arithmetic. W. A. Lay ("Anschauungs und Gedächtnistypen"), has shown that a person may be visual minded in subjects like language or the social sciences and nevertheless be motor minded in arithmetic or geography. Heretofore psychologists taught that a visual-minded person has greater visual acuity and visual efficiency for every variety of experience. Springer, experimenting on 500 children, concludes that visual motor appeals, "see-write," give the best results in all grades except the first and the fifth year; that in the first grade the auditory appeal proved most effective; that in the fifth, the visual. It would seem, therefore, that the best appeal for the initial stages in arithmetic is the auditory; for all other grades, the visual motor, the "see-write" or "see-do." This conclusion was borne out when additional tests were made with numbers of varying difficulties and with children of varying degrees of proficiency in arithmetic. But no general conclusion can be drawn from this one study. It may serve, however, to disturb the smug advice that is so often repeated, "Appeal to all senses." Multiple sense appeal may not be sound, for it may hinder the child of pure sense type. The auditory-

minded pupil may be retarded in his work by persistent visual motor appeals. But this possibility may be offset by the fact that most people are of mixed mental type and are not hurt by a special sense appeal. The atypical child, either the subnormal or the precocious, is frequently a pure type, depending on one sense—visual or auditory or motor—markedly more than on any other. Here, then, is a department of teaching in which pedagogy must suspend judgment and hold itself in readiness to modify its methods as soon as scientific evidences warrant new conclusions.

These researches, though tentative, have a direct bearing on the question of the advisability of postponing the teaching of formal arithmetic until the child is ten years old. No conclusive evidence has as yet been obtained, but the impartial student of educational matters is led to question the amount of arithmetic that is taught and the wisdom of prevailing methods. Many investigations point to the belief that the number instinct in children of seven is often strong enough to warrant the introduction of formal arithmetic and the inculcation of certain basic number habits. It is to be hoped that these experimental investigations will dedicate themselves to the task of supplying a truly scientific basis for methods of teaching arithmetic.

CHAPTER V

THE ORGANIZATION OF A RECITATION IN ARITHMETIC

THE previous chapters concerned themselves with values of arithmetic, courses of study and general principles of teaching the various branches of this subject. We must now turn to a consideration of the effective organization of a recitation period in arithmetic.

VARIETY OF RECITATIONS

The Inductive Recitation.—The traditional recitation with its five formal steps, preparation, presentation, comparison, generalization and application, still serves as a basis for many effective lessons in arithmetic. A first lesson on the reduction of a common fraction to lowest terms, in which children are taught to divide numerator and denominator by the same number, will serve to illustrate this form of recitation.

Preparation: Elicit that 90 cents and 9 dimes are the same; that 90 cents is $\frac{90}{100}$ of a dollar and 9 dimes is $\frac{9}{10}$ of a dollar; hence $\$\frac{90}{100} = \$\frac{9}{10}$; in similar manner, using 25 cents, $\$\frac{25}{100} = \$\frac{1}{4}$. Lead children to conclude that since the values are the same, it would be advisable to use smaller fractional forms in all solutions. Question children on what should be the next lesson in arithmetic. The answer "How to change large fractions into smaller ones" is readily obtained. Note that the lesson has an *aim*, a *motive*, and an *apperceptive basis*.

Presentation: Diagrams like the following are presented.

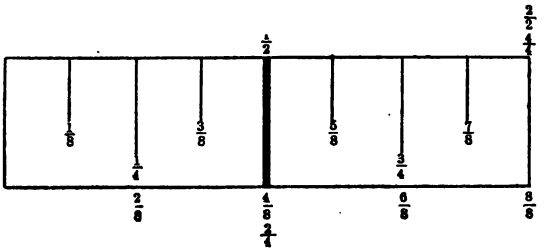


FIG. 5.

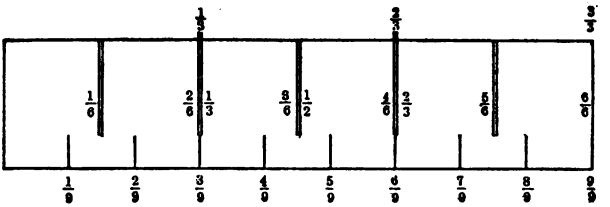


FIG. 6.

The effectiveness of these diagrams can be enhanced by using a variety of colors to indicate the different fractional parts and by having children fold paper to reproduce these subdivisions. With the aid of the diagrams, elicit from the children the following:

$\frac{2}{4}=\frac{1}{2}$ $\quad\frac{2}{6}=\frac{1}{3}$

$\frac{4}{8}=\frac{2}{4}$ $\quad\frac{4}{6}=\frac{2}{3}$

$\frac{4}{8}=\frac{1}{2}$ $\quad\frac{3}{6}=\frac{1}{2}$

$\frac{6}{8}=\frac{3}{4}$ $\quad\frac{3}{9}=\frac{1}{3}$

$\frac{6}{9}=\frac{2}{3}$

Take the first fractional equivalents, $\frac{2}{4}=\frac{1}{2}$, and submit $\frac{2}{4}$ to an analysis like the following: by what must I divide the 2 to obtain 1; the 4 to obtain 2? The third, $\frac{4}{8}=\frac{1}{2}$; by what must I divide the 4 to obtain 1; the 8 to obtain 2? In the same manner analyze each equation and the blackboard work now has the following appearance:

$$\frac{2+2}{4+2}=\frac{1}{2} \qquad \frac{2+2}{6+2}=\frac{1}{3}$$
$$\frac{4+2}{8+2}=\frac{2}{4} \qquad \frac{4+2}{6+2}=\frac{2}{3}$$
$$\frac{4+4}{8+4}=\frac{1}{2} \qquad \frac{3+3}{6+3}=\frac{1}{2}$$
$$\frac{6+2}{8+2}=\frac{3}{4} \qquad \frac{3+3}{9+3}=\frac{1}{3}$$
$$\frac{6+3}{9+3}=\frac{2}{3}$$

Comparison: Lead children to note that 2, 3, 4, or any number may be used as divisor; that where 2 or 3 or 4 is used to divide numerator, the same number is used to divide denominator.

Generalization: Children now attempt to formulate the law. Teacher modifies their language as may be deemed necessary.

Application: Variety of examples now given for oral and for written solution until habit is formed.

If the reader will follow the lesson without noting that it was planned according to formal steps, the entire development seems natural. The formal steps tend to put the burden of the work on the children, to lead them to discover the various relationships among numbers and the final law involved, to give them opportunity for application, and to assure systematic and psychological organization. This method-whole has been much maligned and is, therefore, no longer in pedagogical fashion. But for want of a better substitute, it may still be used with advantage when inductive teaching is desirable. Much of the odium connected with the form of the recitation is due to the misconception that none of these steps can be combined with another or eliminated if it serves no useful function in a given lesson. The formal steps constitute a suggestive plan, not a fixed procedure.

The Deductive Recitation.—A deductive recitation is not devoid of plan and purpose. In the discussion of the place of induction and deduction in the teaching of arithmetic, conditions were listed which make the second mode of instruction desirable. The deductive recitation, in its

complete form, consists of five steps, viz.: the preparation; the data; the generalization; the inference, or explanation, or application; the verification. Whether the recitation is inductive or deductive there must be a preparatory step which calls up necessary old knowledge, establishes an aim, and attempts to stir a motive for the new knowledge. The function of each of these steps can be illustrated by a lesson on the area of a circle.

Preparation: Cite instances necessary to find area of circular rather than rectangular surfaces. Lead children to state object of new lesson. *Aim* and *motive* are now established.

Data: All necessary facts must now be gathered. Review meaning of radius, diameter, circumference and area; have children recall that diameter is twice the radius, and that circumference $= 3\frac{1}{7} \times d$ or πd.

Generalization: The teacher gives the law, namely "$3\frac{1}{7} \times$ radius squared," or πr^2.

Application: Examples in the book are solved by applying this law.

Verification: By a construction exercise the answers can be approximated. Compare answers obtained with those in book. Compare area of circle with area of a circumscribed square.

In teaching long division, the mode of obtaining the least common multiple, or any of the lessons that make the deductive recitation appropriate, these five steps may not all be necessary, nor need they follow the sequence as outlined. It often happens that the data follow the generalization or that the verification is impossible. Any modification, whether of addition, elimination, or combination, may be made, if, in the judgment of the teacher, the effectiveness of a lesson will thereby be enhanced. A lesson without any plan usually entails extravagant waste of time and energy; a plan, modified by the teacher after due deliberation, usually produces a better lesson.

The Recitation Based on the "Steps in Thinking."—The thinking activity is usually started by a need to re-

adjust old experience. So long as life's routine continues without change, reason is discouraged. When a new problem must be solved, a new relation evolved, or contending elements harmonized, the mind's thought processes are set in motion. The child who can multiply by a number of three orders containing any three digits except the cipher finds no need for thought when the multiplier is 246 or 375 or 892; but the moment a zero is introduced in the multiplier, a new condition arises and the mind's power to reason is awakened. Steps, more or less definite in function, are recognizable in an act of reasoning. These are: (1) a conflict in experience; (2) recognition of a problem; (3) attempts at solution; (4) evolving the final explanation or procedure; (5) verifying the conclusion. The scope and function of each of these steps in reasoning can be seen in the following lesson; therefore, no attempt is made to define them formally.

In teaching multiplication by a decimal, certain problems were presented and the children were asked to indicate the mode of solution. They suggested that \$24.54 be multiplied by 8 to obtain the answer for the first and \$72.36 be multiplied by 12, for the second. Since they knew how to multiply dollars and cents by a whole number, they obtained the answers with no trouble. The numbers 8 and 12 were then changed to .8 and .12 respectively, and the children were again asked to indicate the mode of solution. It was clear to them that \$24.54×.8 and \$72.36×.12 would give the required answers. They were asked to perform the operations. Some children stopped at once; others continued beyond the initial point, but all failed to complete the process. The various stages of incompleteness are shown below:

\$24.54	\$24.54	\$72.36	\$72.36
.8	.8	.12	.12
	\$19632		\$86832

A few questions elicited from the children that these multiplications involved a new condition and that the task for the day's arithmetic period must be "to discover how to multiply by a decimal." We have here (1) a conflict in experience, and (2) the recognition of a definite problem.

"What are probable ways of obtaining the answer?" the teacher asked. All agreed that \$24.54 must be multiplied by the eight and \$72.36 by the twelve, but the products would not be the correct answers. The children were then led to conclude that if they knew where to place the decimal points in the numbers \$19632 and \$86832 the entire difficulty would be removed. Each child was then asked to find the place for the decimal point. Some pupils thought it belonged between the 9 and the 6, thus, \$19.632; and between the 8 and 6, thus, \$8.6820; "because if you add the places in the two numbers you get three places in the first example and four in the second." Others objected to adding the places "because these are examples in multiplication, not addition"; these children suggested the answers \$196.32 and \$8.6820, because "in the first example we have two places and one place, $2\times1=2$, but in the second, we have two places and two places, $2\times2=4$." The lesson had, thus far, progressed through the next two steps, viz., (3) attempts at solution, and (4) evolving the final procedure.

The children now realized the need of testing their respective conclusions. Since they knew no method of verification the teacher suggested the solution by common fractions. This was attempted and the following results were obtained:

$$\$24.54\times\frac{\overset{4}{\cancel{8}}}{\underset{5}{\cancel{10}}}=\frac{\$98.16}{5}=\$19.632$$

$$\$72.36\times\frac{12}{100}=\frac{\$868.32}{100}=\$8.6832$$

It was evident that those who " added the places in the two numbers " were correct. After two more examples were solved and the answers tested, the law for pointing off in decimal multiplication was established beyond question.

The caution that was urged in connection with the use of the inductive and the deductive recitation must be repeated here. There is nothing unalterable in the sequence or in the number of these steps. Teachers must rise above the method if it is to serve classroom needs most effectively. Each of these recitations is no artificial organization that has been evolved by teachers of methodology, but it is rather an attempt to incorporate in formal instruction the psychological processes that function in normal and natural life.

Characteristics of an Effective Recitation.—Since each of these forms of recitation may be modified to meet special instructional exigencies, we must formulate the most essential characteristics of any effective lesson in arithmetic. These are summed up in the following questions:

1. Does the lesson begin at the " point of contact " in the child's life and interests? Is the lesson motivated through an appeal to need or to instinctive craving?
2. Does the lesson begin in the knowledge that the pupils have of kindred subject matter? Is the new lesson an outgrowth of old knowledge?
3. Does a clearly defined aim determine the progress of the lesson? Do children as well as teacher constantly keep the aim in view?
4. Does the lesson follow a logical plan, a recitation plan?
5. Is the lesson based on objective experience?
6. Is the child's self-activity stirred and directed at all times?
7. Is there sufficient and varied oral and written drill in the mechanical phase of the lesson, and rich application of the thought phase to problems of social life?
8. Is there an adjustment of the lesson to the needs and abilities of groups of children or do all pupils receive the same instruction, the same drill, the same application?

9. Is the lesson so organized that all sources of waste are eliminated?
10. Is there a well-planned test to ascertain the children's proficiency in the lesson?

THE CONTENT OF THE RECITATION IN ARITHMETIC

As a general rule there must be a division of the recitation in arithmetic into two parts, one being devoted to oral work and the other to written applications. These two subperiods differ in purpose, in spirit, and in method, and must, therefore, be subjected to careful analysis.

The Period of Oral Arithmetic.—*Duration of Oral Arithmetic.*—No day's work in arithmetic must be without its oral period. Oral arithmetic should be assured of a minimum of about one-third of the total time given to arithmetic. Frequently it is given a much greater proportion of the lesson; on days when a new process is taught it usually takes the entire period devoted to arithmetic. From the time which the oral work is permitted to consume, we may deduce its vital importance in developing efficiency in arithmetic.

Aims of Oral Arithmetic.—1. The first purpose of the oral period is *to prepare* the child *for the rapid oral arithmetic prevalent in the business world.* The merchant, the buyer, the broker, the mechanic offering a tentative estimate, the retail clerk or salesman—all these usually work under conditions that make slow written solutions impossible. These people must arrive at their conclusions, accurate or approximate, rapidly and without recourse to paper and pencil.

2. A second aim of oral arithmetic is purely *disciplinary—to train in rapid thinking and concentration.* Written arithmetic affords opportunity for slow, dispirited work. Drill in rapid oral work gives tone and spirit to the entire lesson in arithmetic and makes its results apparent in the written

exercises that follow it. Proficiency in written work is no guaranty of equal or approximate proficiency in oral work, but proficiency in oral work contributes directly to greater accuracy and speed in written work. In the interest of written arithmetic, oral arithmetic must be emphasized.

3. *Continuous drill in the fundamental processes* is the third aim of oral arithmetic. Here emphasis is laid on the purely utilitarian aspect of the subject. All normal children can be brought up to a useful degree of proficiency in the mechanics of arithmetic. At least five minutes a day must be devoted to rapid addition, subtraction, multiplication and division of whole numbers. All the devices suggested in the chapters on the Fundamental Operations must be utilized in daily drills in the earnest endeavor to bring children to the degree of accuracy and speed demanded by the business world. If the school fails in this respect, it fails hopelessly to meet the primary needs of the employer. The plan book of a certain sixth grade teacher emphasized the mechanics of arithmetic by tabulating part of the work of the week as follows:

Oral Arithmetic—The fundamental operations. Daily, 10 minutes.

(*a*) Daily—Rapid column addition, adding statistics found in appendix in geography. Rapid subtraction of type "27 from 82." Rapid multiplication and division by usual circular devices. Time, 5 minutes. And,

(*b*) Monday—Addition and subtraction of common fractions.
Tuesday—Multiplication and division of common fractions.
Wednesday—Four operations in decimals.
Thursday—Finding 1%, 2%, 2½%, 3%, 3½%, etc., of a number. Time, 5 minutes.

Each day ten minutes were devoted to the mechanics of arithmetic, five minutes every day to whole numbers, and five minutes to a more specialized drill in the mechanics of the fractions and percentage. Examples like $.2 \times .1$, $.2 \times .01$, $.2 \div .2$, $.2 \div .02$, $.02 \div .2$, 3½% of \$250, 1% of \$500, 1% of \$50,

1% of \$5, $\frac{1}{2}-\frac{1}{4}$, $\frac{3}{4}-\frac{3}{8}$, $\frac{1}{2}\times\frac{3}{8}$, $\frac{3}{8}\times\frac{4}{5}$, etc., were solved orally with a degree of accuracy and speed that hardly seemed possible at the beginning of the term.

4. Another function of the period of oral arithmetic is *to introduce and develop the new topic.* It is obvious that on a day when a new process is taught the entire period in arithmetic is given almost exclusively to oral work. After the new lesson is understood it is usually applied orally until the mode of thought is well grasped.

How Shall Oral Arithmetic Be Given to the Class?—Spirit is infused into oral work by the alertness which the teacher displays in giving problems and exercises to the class. The materials for oral solution must not be read to a class from a book. The teacher must prepare her work, select her numbers, and then give the exercises to the children with seeming spontaneity. To save time, children may be asked to read the examples from the printed page or a mimeographed sheet. But in all such work, time should be limited and children must be held to strict accountability for answers to every exercise. To state a problem or indicate an exercise and then call on a volunteer for the answer discourages attention and concentration. Every child must feel responsible for all answers within the time set by the teacher.

The Period of Written Arithmetic.—That part of the arithmetic lesson known as " written arithmetic " is merely an extension of the application step of the oral work. Nothing new is taught. Effort is now directed to solving problems that come under the type of what was taught but which have numbers so large as to make written record necessary.

Aims of Written Arithmetic.—Throughout the written work three aims must ever be kept in mind: (1) Accuracy of solution and form must be stressed constantly; (2) speed consistent with accurate work must be the next aim; (3) proper algorism, neatness, logical and sequential ar-

rangement of the various parts of a problem—these constitute the third group of habits that must result from written work.

Written Arithmetic a Collective Record.—The common practice is to use a new sheet of paper each day for the written exercises in arithmetic. If kept by the children these sheets are lost; if stored away by the teacher, they serve no pedagogical purpose. The written work must be kept in a special notebook or in a large envelope in which the daily sheets are arranged in chronological sequence. A collective record shows at once progress or retrogression of any one child in accuracy, neatness, care and effort. The child whose work is improving, experiences a sense of pride in his accomplishment and feels the thrill of encouragement, while his less earnest neighbor is constantly confronted by evidence of decreasing merit. The supervisor who sees a set of arithmetic papers in the middle of a term has no just basis for a judgment of the teacher's work or the progress of the class. Assuming that the class average is forty per cent, what is the inference? That the children have made no progress? That the teacher is weak in instructional power? What if the first week's results show a class average of only twenty-eight per cent? On the other hand, let us assume that a class scores an average of seventy-five per cent in the middle of the term. Are we justified in drawing complimentary conclusions about the teacher's ability and the progress of the class? What if the pupils averaged eighty per cent during the preceding term? A cumulative record may show that although a pupil has scored only fifty or fifty-five per cent during the last fortnight, that same pupil never attained a rating above thirty per cent at the beginning of the term. Cumulative records of pupils' work show progress, retrogression, or stationary standing, and, therefore, afford rich opportunity for constructive criticism and supervision.

HOW TO SECURE MAXIMUM VALUE FROM THE RECITATION

Waste in the Recitation.—It is not an uncommon experience for the observer of classroom instruction to find the recitation in arithmetic opened by the assignment of a problem The children set at the solution with avidity. After three minutes of work children begin putting down their pencils and indicate by exaggerated posture that they have the answer. They sit in idleness waiting for their slower classmates. After seven minutes have elapsed, all pencils are ordered down and a pupil proceeds to the blackboard to make a copy of his work and explain the solution. Questions are asked of backward children and weak points are bolstered up. At least five or six minutes are required for this task. Division monitors are now given the signal to mark the children in their section. This done, the first example is completed at a total time cost of about fifteen minutes. A second problem is treated in much the same way and the half-hour devoted to written work is at an end. But what waste! In such a class the author counted eighteen out of forty children, who solved both problems correctly in a maximum of six minutes. These children lost twenty-four minutes waiting for their slower classmates and were subjected to the stultification of listening to explanations of processes well understood by them. We must now turn to the problem of economy in the recitation, the causes of waste, and remedial suggestions.

Causes of Waste in the Recitation	*Remedial Measures*
1. Time is often lost because of incompetent class management. A considerable portion of the recitation is taken up by distributing material and ruling papers and	1. A monitorial system must be devised that will permit the distribution of necessary material before the class arrives. Children should be encouraged to rule papers before the lesson begins. The rulings must be uniform and exceedingly simple so that the time necessary

preparing elaborate heading. Teachers often force children to sit idly by as they draw elaborate diagrams or write lists of figures on the blackboard. Such time costs are inexcusable.

to prepare the paper or the sheet is negligible. Necessary diagrams, figures arranged for drill, etc., must be made before the class assembles, on large cardboards or on slate cloth, and hung, ready for class use. If unexpected board work must be done, children should be set to solving a problem while the teacher is thus occupied.

2. The dicatation of problems with unnecessary phrases is another practice that makes for waste in the recitation.

2. (*a*) The obvious remedy—to omit all unnecessary phrases—is not sound. Children must be taught to listen intelligently and discriminately and then put down the essential elements. No instructor of law would edit the case for his student. It is hoped that the pupil will develop, in the course of his legal training, the power of selective judgment. The child must be taught to take down problems discriminatingly.

(*b*) Examples should be worked from books as far as possible.

(*c*) Mimeographed sheets containing suitable examples can be prepared and distributed among the children.

(*d*) Examples may be written on cards. These are distributed and the children sent to the blackboard.

3. Time is often lavishly spent working out examples with long numbers and difficult combinations.

3. (*a*) If the purpose is to give drill in the fundamental operations, a mechanical or abstract exercise should be given, e.g., 3426×786; but when problems are assigned the aim is to develop thought in arithmetical situations. The numbers chosen must be such as to avoid arduous manipulations of division and multiplication. This

suggestion does not encourage ultrasimplification of arithmetic, but rather seeks to conserve a maximum of time for the rational phase of arithmetic.

(*b*) *Visualize the Process.*—When an important and difficult type is taught, effective drill in the mode of solution can be given by omitting the mechanical operations altogether and merely visualizing the solution. Let us assume that the new type is in unitary analysis, to find the cost of x units, knowing the cost of y units, the numbers all in fractional forms. A mimeographed sheet containing about twenty-five of these problems, all well graded, is distributed. The exercise reads as follows:

If $3\frac{1}{2}$ yards cost \$10.50, what will $5\frac{1}{4}$ yards cost?

What must I pay for $18\frac{1}{2}$ yards at the rate of $5\frac{1}{2}$ yards for $\$7\frac{3}{4}$?

Paid \$17.25 for $8\frac{1}{4}$ rolls of cloth. Find the cost of $12\frac{1}{2}$ rolls. Etc.

The children are told to read the first problem; the recitation then follows: "Pupil A, what is given? B, what is to be found? C, what is the first step? D, what is the second step? E, how do you perform the first step? F, approximate the answer. G, how do you perform the second step? H, approximate the answer. J, how would you test the accuracy of your final answer?" Each problem is subjected to oral analysis, to visualization of its process, to an approximation of its results, and to a mode of checking, but not a single step

is worked out on paper. In twenty minutes at least twenty exercises are performed and the class has had vigorous drill in the rationale of this type of problem. A few of these problems are given for homework and as many others as may be necessary are solved the next day in class. As a daily mode of instruction, this process of visualization is not to be recommended.

4. Time is wasted for some pupils because work is not apportioned according to ability. To assign a set number of examples and hold all children responsible for the solution of the same number, involves waste for the abler children.

4. Children must be made to feel that they are expected to work at maximum capacity. It is advantageous, therefore, to group children into sections A, B and C, according to their capabilities in arithmetic. Groups A, B, C, are given examples 1 and 2; group B is given an additional example; group A is given two additional examples. In other words, group A is expected to do examples, 1, 2, 3, and 4, while group B does 1, 2, and 3; group C is not expected to do more than the two examples. Examples 1 and 2 are type forms, example 3 is a slight variation, while example 4 requires the exercise of ingenuity for its solution. Work is not only apportioned in quantity and quality, according to ability, but all children are assured of practice and test in the type form. Examples 1 and 2 are explained more slowly. While children in Group C are correcting their errors, those in A and B who have incorrect answers to examples 3 and 4 attend to the explanations; the others, meantime, are set to profitable work.

The same end can be attained by

sending all children in Group C to the blackboard while those in B and in A are in their seats. Work is assigned as above or the problems given the C children at the board may be entirely different from those done by the A and B pupils. The teacher gives the first part of the period exclusively to the children at the board. While the C children correct their errors, the teacher takes up the work with the A and B groups. Any variation which seeks to assign work according to capacity and break the lockstep system in instruction will be found superior to the old system of uniform assignment and uniform requirement of all children.

5. A recitation without purpose or aim is a means of dissipating energy and wasting time.

5. In planning a recitation there must be a very specific aim governing the entire period and determining the nature and the quality of work assigned. To fill the period with problems or exercises is not enough. The purpose may be to increase speed, to emphasize accuracy in mechanical operations, or to develop mastery of a given type of solution. Every step in the recitation and every exercise assigned to the class must be designed to achieve the specific aim; any divergence from this aim results in direct loss of energy and waste of time.

6. Overdrill entails as much waste and produces as much inefficiency as insufficient drill.

6. In an earlier discussion on drill and habit formation, the dangers of overdrill were emphasized. But inordinate drill is found not only in the mechanical, but also in the rational phases of arithmetic as well. Care must be taken not

to overemphasize the type in any branch of arithmetic; the type must be varied after the initial drill periods. So, too, teachers must not indulge in formal analysis of problems, or in writing complete statements when the process is understood by the class.

7. Indefiniteness of assignment is a serious cause of time waste.

7. Assignment in arithmetic, whether for classwork or homework, must be analyzed carefully by the teacher. We must be certain that unnecessary difficulties are not allowed to creep in, that the language and the conditions set forth in the problem are not beyond the pale of the child's experience, and that the problems are applications of principles taught in the class. Improperly assigned work robs children of valuable time and leads to resentment or unwarranted discouragement.

8. The attention of the entire class need not be directed to explanations needed by a minority of the class.

8. After a problem has been completed, those children who made mistakes should be called to the board in a group and the example in question explained to them while the rest continue with the next problem. The second example is treated in the same way; only those who failed in the solution are given the explanation while the others are permitted to invest their time to greater profit in additional work in arithmetic or in some form of study work in other subjects.

9. Failure to use short cuts entails unnecessary expenditures of time.

9. In framing problems the teacher should so select the numbers for multiplication or division as to permit the use of short cuts. Instead of asking for the area to be paved if a gutter measures

54 feet × 129 feet, change the numbers to 54×125 and thus allow the child to add three ciphers and divide by 8. If the short cut is to be of any value, its use must become a matter of habit. The common practice by which children perform their manipulations by the long process and use the short cut only when the teacher says, " Be sure to do this by the short way or you will be marked wrong," is an utter waste. Short cuts must be so taught that children will automatically use them. Then and only then are they sources of economy of time and labor. In the later discussion of short cuts this desideratum is explained more fully.

10. Another cause of waste in a recitation is the assignment of problems that are outside the scope of practical business usage.

10. It was emphasized in our earlier discussion that problems must be real; that the conditions they set forth must make solution seem necessary; that they must constantly reflect the practices of commerce and industry. To violate this cardinal rule is to reduce the recitation in arithmetic to puzzle solution and to divorce school work from the needs of life.

HOME WORK AND PROFICIENCY IN ARITHMETIC

Need for Homework in Arithmetic.—It has not yet been demonstrated experimentally that homework in arithmetic adds to the efficiency which children develop in class work nor what proportional part of a child's efficiency is due to the homework assigned for a given period. It will be necessary to keep careful records of progress and make scientific tests of two sets of children in the same schools, taught by the same teacher under approximately similar conditions,

the one required to do homework, the other relieved from all arithmetical preparation after school hours. It may be that the traditional claims for homework will receive a ruthless jar from the results of such an investigation. Today, homework is justified by teachers on the ground of limited opportunity for drill and application in class, and the presence of ungraded pupils. For these reasons, teachers allow the work of the class to encroach on the child's free time.

Difficulties Caused by Homework.—Every teacher must meet additional disciplinary and instructional problems which come as by-products of homework. Some children do not carry out the assignments—investigation is necessary and punishment must be meted out. Others perform their tasks carelessly and imperfectly—measures must be taken to correct these pupils. Most of the problems which arise from homework can be obviated or minimized by careful attention to correct assignment of these exercises. We must, therefore, turn to a consideration of proper assignment.

How to Assign Homework Properly.—1. *Homework Must Be Supplementary.*—In all but the last school year, homework must be based on what was taught in class. Homework becomes an extension of the application step of the lesson or a specialized drill period. At the end of the school course, new material may be assigned for home study because the aim is to test children's power to derive knowledge from the printed page as well as to give added opportunity for application.

2. *Homework Must Be Assigned with Careful Attention to Time.*—Homework must be given sparingly in all classes. Most authorities agree that homework must not be given in the first three years, for with these young children it may work irreparable harm rather than good. In the early grades, children must work under supervision so that no

incorrect habits of form, of content, or of manipulation will develop. Through about the fifth year, no problems should be assigned in homework. Children should be required to solve examples in fundamental operations and type forms, the time cost of which can be approximately determined. In no grade in the elementary school should homework in arithmetic take more than thirty minutes of the child's time. More progress is to be expected from systematic class drills and supervised study periods than from excessive homework.

3. *Homework Must Be Assigned with Careful Attention to Difficulties Involved.*—Every example assigned must be examined to see that there are no difficulties of language or superfluous numbers. Unnecessary difficulties must be removed. It is advisable to assign homework at the end of the arithmetic period in order that children may read the examples and ask such questions as they deem necessary.

4. *Homework Must Be Assigned with a View to Correction.*—A primary consideration which determines the assignment of homework is the matter of inspection and correction of children's products. Homework which is not examined soon degenerates and becomes a source of demoralizing influences in all written work. Some teachers have the correct homework put on the board and each child marks his neighbor's results. The children who have any example incorrectly worked are helped, during spare moments, by those who put the work on the board correctly. The books are then collected and are inspected by the teacher, who usually puts some mark on the day's work. In this way each child sees his error, learns the correct solution and receives the teacher's comment—all at the expense of little time and effort on the part of the teacher. Any simple system which guarantees inspection, correction and help may be substituted for the device suggested.

5. *Homework Should Be Kept in Notebooks.*—The reasons advanced for keeping the daily work of arithmetic in a notebook rather than on separate sheets of paper apply with equal force to matters of homework. It may be possible to keep classwork in the first half of the book and the homework in the second. Whatever the plan, the aim is to have a cumulative result which shows at a glance the status of each pupil's proficiency day by day.

6. *The Pupil Must Be Assured of a Favorable Place for Study and Homework.*—Teachers must ascertain whether their pupils can secure conditions favorable for study and homework. Children often fail to carry out the assignments for homework because their homes make it impossible for them to retire to a place that is quiet. A classroom in charge of a teacher should be kept open after school hours so that children may be assured of a quiet place, and the necessary equipment of pen, ink, blotter, dictionary, simple reference books, and sympathetic help.

7. *All Pupils Not to Have Same Homework Assignment.*—If we look upon homework as an added application of knowledge taught in class, it is apparent that, in most cases, all children are not in need of the same amount of drill in the same subjects. Homework must be differentiated so that pupils proficient in mechanics of arithmetic are required to work problems, while those deficient in fundamental operations have more mechanical exercises. To require all children to perform the same homework must result in overdrill for some and insufficient drill for others.

8. *Not All Homework Need Be Written.*—The usual form of homework in arithmetic must necessarily be written, but it need not be the unvarying form. If tables are assigned for home study, it is not advisable to require all children to write these a set number of times. Those who master the combinations or the equivalents from class drill may be excused from further drill. Children should be encour-

aged to study their own characteristics, to note whether they are visual, auditory, or motor minded, and then appeal to the sense that seems most active for them. To require auditory children to devote time and labor to writing the tables ten times may retard progress. Teachers must learn to accept any evidence of earnest attempt to study the table in question. Some children may bring in written evidence, others notes, written by themselves, and signed by their parents, telling the teacher: "This is to certify that Master William Smith studied the table of linear measure for ten minutes by looking at it and saying it over and over." In this differentiated homework the child is, therefore, enabled to work along the line of his native endowment and to discover those personal idiosyncrasies which make for efficient study.

Conclusion.—Much has been said of the value of homework for developing concentration, self-reliance, habits of study, and for bringing parents into closer relation with the work of the school. The powers enumerated above can be developed by the school through other agencies which do not encroach on the child's hours of free play. And it is to be hoped that the school will evolve more social means than homework of bringing parents into intimate contact with the school.

GROUP TEACHING IN ARITHMETIC

The Need of Group Teaching in Arithmetic.—In our earlier discussions, it was shown that arithmetic is a complex of processes and functions of varying characteristics. Proficiency in one branch of arithmetic offers no assurance of approximately equal proficiency in another branch of the subject. It was also shown that there is no general arithmetical ability. A child may develop power quickly in one phase of arithmetic and experience no little difficulty

in attempting to master another. A small number of pupils respond to a rational appeal when a new topic is presented, some to an objective presentation, while others see the reason governing the *modus operandi* only after using the new process again and again. Experimental evidence seems to show that native endowment is the most telling factor in determining proficiency in arithmetic. Mathematical judgment of younger children with less experience and training in arithmetic often surpasses that of older ones with three or four years more study of the subject. The inference seems conclusive that arithmetic demands group rather than class teaching, special forms of presentation and specialized forms of drill to approximate the needs and the capacities of the pupils.

Devices for Group Teaching in Arithmetic.—It is not within the scope of this book to treat of the devices and methods of class management which are necessary for successful group teaching. Books on class management and on group teaching give ample discussion of this phase of our problem. We must stop long enough to indicate the nature of the differentiated work.

Specialized Drills.—Since drill must be adjusted to the need of the pupil, specialized tests must be given at regular intervals to discover the degree of proficiency which each pupil has attained. These tests will indicate individual weaknesses and serve as a basis for prescribing the form of drill calculated to contribute most to a child's needs. Through differentiated homework, differentiated class drills, differentiated written work in class, children may be helped to overcome their peculiar weaknesses and limitations.

Differentiated Teaching.—In addition to differentiated drills, there must be differentiated modes of presenting new lessons. For the slower children, the lesson must be more objective, graded with greater care, less rationalized; for the abler pupils, a shorter presentation with a greater ap-

peal to the logic of successive steps may prove more effective. With the first group, the work may touch only general principles; with the second, a more intensive appeal may bring gratifying response. A uniform mode of instruction, however skillful, is not calculated to evoke a universal response.

Monitorial Teaching.—A common classroom practice is the appointment of abler pupils as special coaches of their less proficient classmates. The benefits which the teaching pupil derives are ethical rather than mathematical. These are often counterbalanced by exaggerated estimates of ability and superior attitudes which are developed by these aids. The backward pupil is not taught, he is "shown how," he is coached, he is forced to imitate. We have conveniently assumed that this system of class coaching will lift the arithmetically submerged to the necessary level of proficiency. Nevertheless we reject the conclusion that with slower pupils the teacher should not teach but merely present each fact or process through purely mechanical imitation. The reader must not conclude therefore that student coaches must not be appointed; he must realize that we have not yet justified the practice of intrusting to children the highly skilled art of teaching arithmetic. In all teaching situations where faithful imitation or emulation of conduct is the final goal, properly supervised monitorial teaching has its place.

CHAPTER VI

TEACHING NUMBER CONCEPT; COUNTING AND MEASURING

THE concept "number" is foundational in the study of methods of teaching arithmetic. There is a variety of interpretations of this basic term, each determining in its own way a definite mode of instruction. It is the function of this chapter to set forth the meaning, the origin, and the development of the number idea and to formulate a system of teaching number which incorporates the lessons of modern experimental psychology. With this aim in view we shall attempt to summarize four important conceptions of number: (1) number as a symbol; (2) number as a sense fact; (3) number as a rhythmic idea; (4) number as an idea of relationship resulting from a variety of thought processes. Each of these interpretations of number will determine, very definitely, a distinctive mode of teaching.

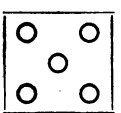
I. NUMBER AS A SYMBOL

The earliest conception of number which governed teachers held that number is merely a symbol for a group. Arithmetic was a purely utilitarian subject which summed up all number symbols and rules governing their manipulation. The *figure* 5 was made synonymous with the *number five* and was taught by a severely deductive process based on imitation and endless repetition. The arithmetician was an artisan and the teachers of arithmetic were incorporated in the society *Rechnenmeisters*. Since number was a symbol, no educational influence was sought from the teaching of arithmetic. Although this explanation

of number is old, it is not necessarily obsolete, for much of the deductive teaching with its purely imitative characteristic gives evidence that number and symbol are still identical in the minds of some teachers.

II. NUMBER AS A SENSE FACT

Number the Result of Sense Experience.—Pestalozzi was the first educator who succeeded in supplying a psychological basis for number. To him, number was more than a symbol; it was a mental image derived from sense experience. Four books are seen, four coins are touched, four sounds are heard—these sense impressions are recorded in the brain, and the mind, by a one to one correspondence, becomes conscious of *four*. Just as the facts *red*, *rough*, *warm*, *circle* are acquired through the senses, so, too, number arises from a repetition of sense experience.

Conclusions for Teaching.—The pedagogical conclusions evolved by Pestalozzi give greater clearness to his conception of number as a sense fact. His first maxim for teachers is "*Number before figure.*" He differentiated clearly between | . . . . . | or | ///// | and the symbol **5**. The mental picture which images the group of dots or lines is the number; the written symbol, **5**, is the figure. To teach the figure **5** before the picture | . . . . . | or | ///// | or  is clearly imaged in the mind, is to teach a sign before the object which it represents.

A second pedagogical practice which Pestalozzi made famous was, "*All number must be taught orally and objectively.*" If number is a sense fact, children must learn number by seeing, hearing, and touching every variety of illustrative aid—papers, beads, strings, chairs, pupils,

taps, bell sounds, hand claps, etc. To insure this perceptual basis for number and guard against too early an introduction of figures, he emphasizes, for the first time, oral teaching of number.

Pestalozzi observed that counting, for some reason, appealed to children as much as manipulating concrete materials. He therefore decides that *all number must be taught through counting* all possible objects and sounds in the environment. In seeking to teach number exclusively through counting, Pestalozzi's theory naturally suffers from the weakness of any extreme position. To Pestalozzi must be given credit for having introduced the perceptual basis in number work and the disciplinary value of arithmetic.

III. NUMBER AND RHYTHM

The Origin of Number in Rhythm.—A conception of number closely allied to that evolved by Pestalozzi is found in G. Stanley Hall's belief that " counting is the rhythmical punctuation of the stream of consciousness." The human mind is extremely sensitive to rhythmic repetitions and is agreeably affected by rhythmic experiences. There is an element of pleasure in keeping time, in regular nodding, in discerning a regular system in the tickings of the clock, and in rocking in regular count. Repetitions of certain numbers are looked upon with favor while others are not attractive. Shopkeepers find that they can sell more articles when the price is set at 47 or 49 cents than at 45 cents, more at 19 cents than at 16 cents. The " 5 and 10-cent " stores or the " 3, 9, and 19-cent " stores have an asset in numbers selected, because these command attention.

How to Teach Number through Rhythm.—If we accept the view that numbers have their origin in rhythm and that " counting is the rhythmical punctuation of the stream of consciousness," the rhythmical series must be the starting point of the lesson. The series, 1, 2, 3, 4, 5, . . . 2, 4,

6, 8, 10 . . . 3, 6, 9, 12, 15, . . . must be seen, heard, done and felt. After attention has been arrested by the series and when there has been sufficient repetition of the series, the number names are given. These names are applied to a series of objects that can be seen and touched, to a series of movements, and finally to a series of sounds, until these names become habit. The third step applies these number names to new number experiences. This method of teaching number insures objective presentation, a thorough perceptual basis, exclusive oral drill and the sequence of number before figure.

IV. NUMBER AS A RATIO RESULTING FROM MEASUREMENT PROCESSES

The Psychological Conception of Number.—An explanation of the origin and function of number which, historically, is the most modern, is the psychological conception, best formulated by McClellan and Dewey in "The Psychology of Number." This theory makes, at least, three basic contributions to our knowledge of number: (1) Number is not a sense fact, but a rich idea; (2) number is ratio; (3) number arises and grows in measurement processes. Each of these ideas is rich in content and practical implications and must, therefore, be submitted to more detailed analysis.

1. *Number Not a Sense Fact, but an Idea Resulting from Thought Processes.*—Pestalozzi, Grube and their followers, lost in the psychology of perception, regarded number as a sense fact. Let a group of things be experienced and the mind apprehends number. But a critical analysis will show that merely the presence of a multiplicity of objects before the senses will not account for the idea of number. The savage brings down many animals, but his knowledge of number does not correspond to his success in the hunt. Children who see groups of 5, 12, or 9 sticks do not know the numbers five, twelve or nine. Show chil-

dren the following arrangement of an equal number of equally sized lines and ask them which group contains more:

invariably they select the former. Form, color, size, roughness—these are sense facts; but number is a complex of relationships which arise from a variety of thought activities. Ask the child, who can count blindly to ten, to show you "five" on his fingers and he points to successive fingers as he calls, "one, two, three, four and five." At the word "five" he stops and points to his thumb. "Five" means the fifth, not five one's; not two, two and one; not two and three; not two and three one's; not three and two one's; not one and four.

That number is not a sense fact can be deduced from other observations. Children who know the number names up to ten and who are asked to count ten concrete things usually run ahead of the objects in applying the number names. If two or three of these objects are surreptitiously removed, they are not missed by the pupils. Number is learned not by looking at or touching ten objects, but by analyzing the group of ten, comparing the group of ten with one of two or of five. *Only as the mind goes through the thought activities of comparison, of contrasting, of measuring, of analyzing and of synthesizing, will the complex idea of number arise. Number is an idea which the mind evolves and applies to objects, not an idea which is extracted from objects.* Number is a complex interpretation of sense experience, not a mental image, a photographic reproduction of sense experience.

2. *All Number is Ratio.*—If number is an idea, our next task is to discover the kind of idea. The preceding analysis shows that number is an idea which summarizes the relations that exist in a group; hence every number must be a ratio. Any number expresses the ratio

which exists between a measuring unit and the whole quantity. Thus, ten cents refers to ten times one cent or twice five cents; three yards refers to three times one yard, or three times three feet, or nine times one foot; nine dollars, to the person who saves three dollars a month, is three times three dollars; to the person who saves a dollar a week, it is nine times one dollar. It follows, therefore, that the child does not know the number six when he can recite, "one, two, three, four, five, six." The number six, the idea *six*, is known only as the following ratios and combinations are known, 6×1, 3×2, 2×3, $5+1$, $4+2$, $2+4$, etc. In teaching number as a sense fact through counting, every number was produced by adding one to the preceding number, e.g., to teach eight, review 7 and add 1; to teach nine, review 8 and add 1; to teach six, recall 5 and add 1. This system eliminates the ratio idea. In the ratio interpretation, number does not result merely by adding *one*, by using *one* as the unit, but by employing every possible quantity, two, three, etc., as the measuring unit. The function of number, is, therefore, to sum up ratios.

The Changing Unit of Measure.—A unit of measure is any quantity or any standard used to compare quantities of the same kind. This definition precludes the exclusive use of *one*, unity, as the sole unit of measure. In the numerical experiences of life, the unit of measure is usually not one, but almost any other number. In measuring a length, the carpenter uses a "two-foot" rule; in purchasing eggs, the unit is usually twelve; in purchasing thirty cents' worth at six cents a quart, the unit is six cents; in computing populations the unit is 1000 or 10,000 or 100,000, not 1; in making payments of \$.75 or \$1.25 or \$1.75, the unit is usually \$.25.

The idea that the unit of measure must not be fixed constantly as *one* is helpful in all stages of progress in arithmetic. In examples of the type, "At the rate of 6 for 9 cents what will 12 cost?" the old solution required the finding of

the cost of one, $\frac{1}{3}$ of 9c., and then multiplying that by 12. If *one* need not be the unit, then six is selected in this case; in the example, two units are purchased ($12=6\times2$), hence the cost is 9 cents$\times2$ or 18 cents. Compare the two forms of oral solution in the following example:

At 3 for 5¢, find the cost of 18

Unitary Analysis	*Ratio Method*
$\frac{1}{3}$ of $5¢=\frac{5}{3}¢$	
$\frac{5}{\cancel{3}}\times\overset{6}{\cancel{18}}=30¢$	
or	
$\frac{1}{\cancel{3}}$ of $5¢\times\overset{6}{\cancel{18}}=30¢$	$18=6\times3$, hence cost of 18 is $5¢\times6=30¢$
In this mode of solution, this computation usually necessitates written work.	What is a complex solution by the unitary method is a simple oral problem by ratio method.

If $\frac{7}{8}$ of a yard of lace cost \$2.15, what will $6\frac{1}{8}$ yd. cost?

Unitary Analysis	*Ratio Method*
$\frac{1}{8}$ yd. cost $\frac{1}{7}$ of \$2.15	$6\frac{1}{8}=\frac{49}{8}$
$\frac{8}{8}$ yd. or 1 will cost $\frac{8}{7}$ of \$2.15	$\frac{49}{8}=7\times\frac{7}{8}$
$6\frac{1}{8}$ yd. cost,	
$\frac{\cancel{8}^{1}}{\cancel{7}_{1}}$ of $\$2.15\times\frac{\overset{7}{\cancel{49}}}{\cancel{8}_{1}}=\$15.05$	Cost $\$2.15\times7=\$15.05$
or	
$\frac{1}{8}$ cost $\frac{1}{7}$ of \$2.15	
$\frac{49}{8}$ cost $\frac{1}{7}$ of $\$2.15\times\overset{7}{\cancel{49}}=\$15.05$	
Here again a written solution is usually necessary.	Note again that the complex solution becomes a simple multiplication by 7.

In the problem, "If $\frac{7}{8}$ yd. cost \$2.15, find the cost of $5\frac{7}{8}$ yd., approximate answer orally," the oral steps are:

$5\frac{7}{8}$ yd. $=\frac{47}{8}$ yd. $\frac{42}{8}$ yd. $=7\times\frac{7}{8}$ yd. $\frac{47}{8}$ yd. will cost a little less than $\$2.15\times7$ or a little less than \$15.05.

Without the aid of ratio relationship, approximation, so frequently used by every person in business, becomes impossible without paper and pencil computation.[1]

[1] A method of teaching arithmetic by subjugating every phase to the ratio idea is the Speer System. Speer, Wm. W., "Arithmetic." Ginn & Co. The basic principle of this method is summed up by its author, "The fundamental thing is to induce judgments of relative magnitudes." This is done by presenting geometric forms, lines of varying lengths, objects of varying sizes, etc. The teacher then attempts to induce the child to see relative magnitudes and through these to evolve the combinations in the former fundamental operations, concepts of fractions, decimals, etc. Illustrations of typical problems follow:

"Find edges of different solids and tell whether they are longer or shorter than other edges.

"Make sentences like this: this edge is longer than that one and shorter than this one.

"Find circles. Find circles that are larger than others. Smaller than others. Find the largest. The smallest.

"B is how many A's? (A is a cube, B a prism). What part of B is as large as A? A equals what part of B?

"If one foot of molding costs 17¢, how many 17¢ will 1 yard cost?

"If ½ bbl. lasts 1 month, how long will 1 bbl. last?

"I use 1 yd. ribbon for a hat, ⅔ yd. for collar; how many yards do I use?

"Had 4 horses. Sold ¼ of them. How many did I sell?

"Mary had a qt. of berries and sold a pint. What part of the berries did she sell?

"What is the ratio of the number of cases of fruit which can be bought for $x at $15 each to the number of cases which can be bought for $x at $5 each? Write 3 problems similar to this and state what comparisons you would make in solving them."

The Speer Method holds that teaching is successful when the subject matter arises out of the child's needs; that the child meets varying magnitudes and must learn to estimate their value. If arithmetic is to free the mind from the concrete and give it power to deal with abstract relations, this type of work, Speer holds, is inevitable; but to teach mechanical operations to the exclusion of ratios produces helplessness in advanced arithmetic.

We may sum up the limitations of the Speer Method under the following points:

1. Central thought is good, but when made the sole aim of arith-

KINDS OF UNITS OF MEASURE.—In presenting number as ratio we must distinguish three classes of units. The first is the *undefined unit* which measures by "more or less," by "much or little," by "many or few." The second is the *defined unit*, the pace, the hand, the step, measures that vary within certain limits but which convey an idea of dimension. The third is the *standardized unit*, the improved, unalterable unit like foot, yard, meter, gallon. In the earliest grades the idea of ratio, of magnitudes, must be built up by using the undefined unit. As the child's mind grasps the relationship in varying quantities the defined and the standardized units must gradually be introduced to supplant the cruder notions.

3. *Number Must Result from Measurement as Well as from Counting.*—If we accept the view that number is a richer psychic product than a mere sense fact and that it represents a ratio, then the number idea must grow out of measurement processes. The older conceptions always sought to have number evolve in the mind of the child through counting concrete objects. But counting may resolve itself into a mere application of names in a set sequence, while measuring involves relations, analysis, synthesis—in a word, discovering a ratio. The number idea must, therefore, be induced through a process of measuring quantities with indefinite, definite and standardized units. Before applying this psychological conception of number to actual teaching, we must stop for a further analysis of counting and measuring, for the teacher must have a deeper insight into the processes and functions through which she teaches number.

metic much that is necessary is forced. Addition is forced to arise out of ratios.

2. Business arithmetic and business situations are not emphasized.
3. Ratio idea in fractions is introduced too early.
4. Material is too limited.
5. The application is too narrow.

Counting	*Measuring*
1. An attempt to ascertain "how many" units are repeated.	1. An attempt to ascertain "how much" of a quantity there is.
2. Refers to multitude.	2. Refers to magnitude.
3. Mind passes from unit to unit, hence a synthetic process.	3. Mind estimates the entire quantity by noting its ratio to the unit, hence an analytical process.
4. It gives the child a rhythmic sense in time or space.	4. It gives the child a sense of equivalents, 6 pints = 3 quarts = 24 gills.

An illustration will make these attributes of counting and measuring clearer. Let us assume that the child is required to discover how many "nickels" there are in 20 cents, how many feet in the length of a 4-yard board or how many pints in 3 quarts. In each case the child begins with the entire quantity, 20 cents or 4 yards or 3 quarts; the unit 5 cents or 1 foot or 1 pint is selected. The entire quantity is analyzed in terms of the unit, i.e., measured; the results obtained are 4, 12, and 6 respectively. In the actual measurement the process was analytic, the question "how much" was answered, and the mind discovered equivalents, 20 cents = 4 "nickels," 4 yards = 12 feet, and 3 quarts = 6 pints. But in noting how often the unit, 5-cent piece, or 1 foot, or 1 pint, was repeated, the process used was that of counting; in each case it was synthetic, answered the question "how many" and was rhythmic in its repetitions.

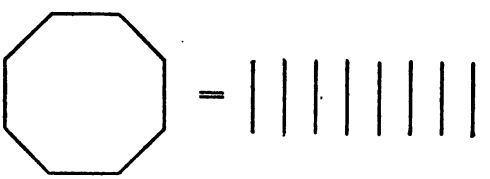
It is to be noted that every act of measurement involves counting, and every completed act of counting results in the measurement of a quantity. McClellan and Dewey sum up this relationship with "Nevertheless all counting is measuring and all measuring is counting." Taken literally, this statement is not true, for these two processes are

not the same; regarded as a summation of final results, it indicates the mutually supplementary relationship existing between counting and measuring.

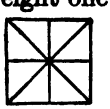
Application of Psychology of Number to Teaching.—This analysis of the theory of number has little worth unless it can be applied to actual teaching. The pedagogical conclusions derived from the notions that number is not a sense fact, but a ratio resulting from measurements, lead us to teach number by following four steps, viz.:

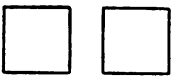
1. Present whole quantity to be measured and give the child a vague idea of the number to be taught.
2. Select the unit or units of measure.
3. Perform the measurement process by counting the repetition of the unit.
4. The result obtained, the ratio between the entire quantity and the unit, is the number.

A. In teaching the number eight, we present eight sticks to the class and ask the pupils to reproduce with these what is made with chalk lines on the board. The first figure would be the following:



Every child first makes the octagon and then takes it apart to discover that eight one's make up eight.

The figure  is made and resolved into eight sticks to lead the class to the same conclusion, eight one's are eight.

B. The next group  gives two squares or boxes. The children are led to find out how many sticks in one box, in two boxes, and finally, how many four's make eight. The necessary drill and application are now given on 4 and 4, two 4's, 8 has how many four's, 4 and how many will give eight, had 8 and lost 4, etc. This second step (*B*), with its variety of drills and applications, may take a number of periods, but time is really gained, for the child is learning addition, subtraction, multiplication, and division combinations in the richest possible associations.

C. The children now set up Indian tents, and are led to discover that there are two sticks in one tent and in four tents, there are four two's or eight. The drill on 2×4, 4×2, $2+2+2+2$, $6+2$, $4+2+2$, $2+6$, $8-2$, $8-4$, $8-6$, etc., must now be given, not in the symbols used here for economy, but in language forms similar to those in step *B*. Here, again, the child is learning fundamentals in rich associations and reciprocal relations. This step alone may afford arithmetic material, abstract drill, practical problems, etc., for a week or two.

D. The eight sticks are now used to make a pentagon and the child is led to discover that 5 sticks in the figure and the 3 that remain, make 8. This yields $5+3$, $3+5$, $8-5$, $8-3$, $5+?=8$, $3+?=8$, etc., another group of combinations well worth mastering in related sequence rather than in isolated tables of addition and then subtraction.

E. The facts in the preceding steps are now subjected to a multiple sense drill, as follows:

(1) Visual Appeal: Children point to pupils, seats, desks, books, window-panes, electric bulbs, etc., to show how many two's in eight, how many four's in eight, 6 and

how many make 8, 5 and how many make 8, 8−3 will leave how many, etc.

(2) Auditory Appeal: Tap with pencil or ruler, ring bell, clap hands and children recognize the combination. The teacher taps xx xx xx xx and the child called on says " eight, four two's." The teacher taps xxxx xxxx and the pupil says " eight, two four's." All the combinations can be reviewed in this way.

(3) Muscular Appeal: Children take steps, hop, sing a note, arrange fellow pupils, tap with pencil, clap hands, etc., to show their knowledge of any combination of eight.

Teacher: " How many two's in eight? "

Pupil: Taps with pencil xx xx xx xx; or arranges eight pupils who are standing in front of the class into four groups of two each.

Teacher: " Five and how many more make eight? "

Pupil: Taps xxx; or arranges the eight fellow pupils into two groups of five and three and points to the group of three.

How Does This Mode of Teaching Number Embody the Psychological Principles Laid Down?—An analysis of the procedure outlined above shows that in each case the new step began with the entire quantity eight; then the unit of measure 1 or 4 or 2 was selected; this was used to measure eight, and the final ratio was discovered. The four steps enumerated at the beginning of the lesson were followed faithfully. The concept " eight " is a ratio, resulting from a variety of measurement processes; the unit was not always one; the lesson did more than present eight objects to the senses; it involved analysis, synthesis, and relating of groups. The number eight, taught by this psychological method, is a rich idea which sums up a host of combinations and associations. When we consider that most urban courses of study require for the first term no more than a knowledge of the four fundamental operations with numbers

from 1 to 10, there can be little reason for objecting to the length of time it would take to teach each of the numbers up to 12 by this method.

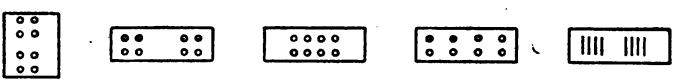
Effectiveness of Teaching Number through Counting v. through Measurement.—The distinctive elements in the method as outlined may be emphasized by comparing the number concept "eight," taught by a process of counting (i.e., eight is identified with 8 objects or with 7 objects and 1 object), with that resulting from a method which emphasizes the measurement process.

Number Result of Counting	*Number the Result of Measurement*
1. Adopts 1 as the constant unit. Eight is taught as 7+1; 5 as 4+1. The new number is the result of adding one to the preceding number.	1. Any quantity is used as a unit and a new number is evolved by measuring the quantity by the unit selected.
2. A mechanical method because all numbers are taught in same way.	2. Every change in unit gives a new combination and a new ratio.
3. A synthetic process.	3. An analytical process.
4. Comparatively little mental activity is called forth in teaching eight as seven and one more. Mere repetition will suffice.	4. Much mental activity is required to discover that eight is four two's, two four's, five and three, etc., as well as seven and one.
5. To sustain interest new objects must be counted. Interest is thus directed to the sense material, not to the numbers, and must therefore be characterized as "fictitious interest."	5. Interest is maintained by changing the combination that gives eight and leading the child to discover the identity between eight, four two's, and two four's. The same objects are used; interest is real, for it is concerned with the ratios in the number.

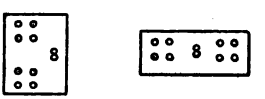
6. The combinations of a number, 6+2, 7+1, 5+3, 4×2, 2×4, etc., are learned by pure rote through incessant repetitions.	6. Much drill is necessary on combinations even in this method, but it is drill following the discovery of the combinations by the child.

The Complete Steps in Teaching Any Number through Twelve.—Thus far, the suggestive lessons on number applied the measurement idea and underemphasized the counting process. It remains to systematize and grade the steps in a lesson on any number through ten or twelve. Numbers beyond twelve must be taught by a counting process; the enrichment of the number ideas will result from later combination when the child learns the tables. How shall the child learn the number " eight "?

1. Let children count one, two, three, . . . seven, eight; one to ten; five to eight; five to ten and further. This mechanical counting gives number names and helps to place " eight " in the number series.
2. Use " eight " with undefined units, 8 disks, 8 children, 8 papers, 8 hand claps, 8 hops, etc.
3. Connect " eight " with number pictures.

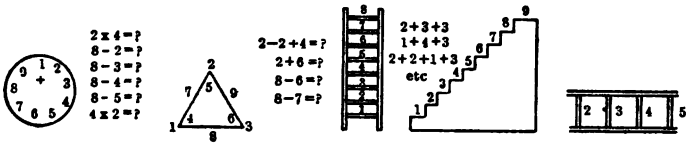


4. Connect " eight " with its symbol and number picture.



At this step the child knows the number name, recognizes the group or quantity " eight," can read the symbol, and has a vague feeling of the place of " eight " in the number series. The following steps refine and enrich the number idea " eight."

5. Introduce the measurement idea and follow the steps as outlined on the number eight in this chapter.
6. Drill on the combinations of "eight," using the following:
   (*a*) Variety of mechanical devices, circle, stairs, ladders, railroad ties, etc.



The teacher, pointing to the figures 5 and 3, calls for the answer; pointing to 3, says "and how much makes," and then points to 8, etc.

(*b*) The algorisms of fundamental operations:

3	8	8	4
+5	−2	+?	×2
		4	?

(*c*) The equation form of drill:

$2\times4=?$ $\quad 4\times2=?$ $\quad 6+2=?$

(*d*) The decomposition form:

$2\times?=8$ $\quad 4+?=8$ $\quad ?+3=8$ $\quad 4\times?=8$

7. Employ the various combinations in practical problems.

   At 2 cents an apple, find the cost of 4 apples.
   A boy who has 3 cents saves 5 more. How much has he?
   Each iron weight weighs two pounds. How many iron weights does the grocer put on the scale to weigh 8 lbs.? etc.

These steps give mastery in oral as well as in written forms, in every variety of combination, in mechanical associations as well as in the applications of problems. The number eight is a rich idea summarizing all the possible relations and ratios in eight, and incidentally teaches the fundamental operations.

The Use of Number Pictures in Teaching Number.—*Their Teaching Value.*—A moot question in teaching number is the use of number pictures referred to in step 3 in the lesson-plan. The number pictures have admitted values, among which may be mentioned: (1) They offer illustrative aids; (2) they appeal to the visual minded; (3) in the child's mind they associate a number with the group it represents; (4) they are simple and can be made on blackboards and cards by pupils as well as by teachers. Experiments by Lay tend to show, almost invariably, that with nearly all children a visual appeal is more effective than an auditory appeal even when sounds are heard in rhythm. It would seem, therefore, a legitimate conclusion to hold that early work should make spacial and muscular rather than temporal appeals. But there are serious limitations in the use of number pictures which must ever be borne in mind: (1) Number pictures aid only in the initial stage of learning; after this, concrete presentations and visualizations retard progress; (2) The ear-minded child finds that this purely visual appeal often hinders comprehension as well as retention. But it must be admitted that few children belong to this pure type, for most individuals are of the mixed sensory type; (3) number pictures do not give rise to the number concept. It was seen that number is the result of measurement processes, not of an association between a number name and a group. These limitations of number pictures argue not for their complete elimination, but rather for their use as objective aids in introducing the number idea.

What Form Shall Number Pictures Take?—Experimental educators are seeking to discover scientifically that form of number pictures which will make the most effective appeal to children. Thus far their investigations tend to establish the following conditions governing form of number pictures:

1. One row pictures are not to be favored; e.g., o , o o , o o o , o o o o , o o o o o , ...

2. Any arbitrary or accidental arrangement is not effective; e.g., o o o / o o / o for 6.

3. In the best arrangement, elements are grouped in two rows, in quadratic form, with equal spaces; e.g.,

• • • • • • • • •
•, •, • •, • •, • • •, • • •,

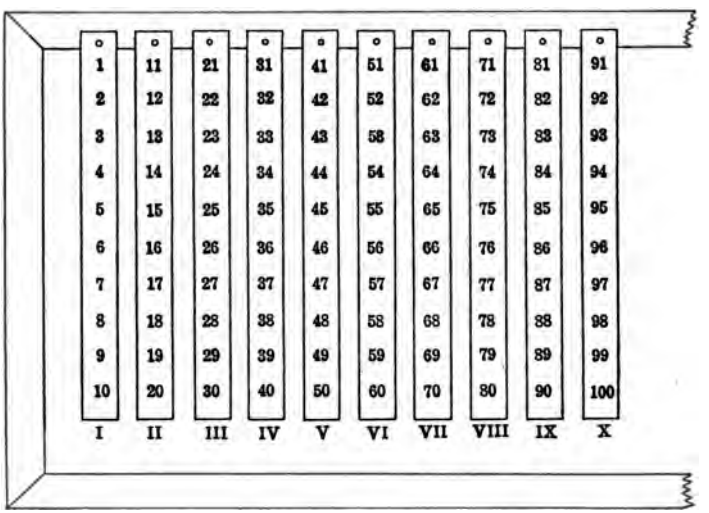


FIG. 7.—Ten Charts Suspended from Blackboard.

For step 1, counting exercise, use Chart I.
For step 2, 3, 4, counting exercise, use Charts I to X inclusive.
For steps 5, 6, 7, make 10 charts (reverse of these) and letter them as follows: I, 100, 110, 120, 130, . . . 200; II, 210, 220, 230, . . . 300; III, 310, 320, 330 . . 400; X, 910, 920, 930, . . . 1000.
These 10 charts suspended from blackboard give the basis for most of the counting drills in primary grades. Count across as well as up and down.

4. The circle of about 5 cm. is superior to the dot, line, or other mark.

5 The dark circle on a light background or light circle

on a dark background seems to give the best results. Card and blackboard presentation are, therefore, effective.

These conclusions are not final nor do the tests have the unqualified approval of the experimenters. But investigations made with care and carried out with all due regard to accuracy seem to point to these five conditions governing the form of number pictures.

Gradation of Counting Work.—*Forms of Counting.*—Success in early counting work is often determined by careful gradation of the various forms of counting. A system of gradation that has proved helpful is here suggested:

1. Counting 1–10 inclusive by 1's.
2. Counting by 10's to 100; e.g., 10, 20, 30, . . . 100.
3. Counting 21 to 30, 31 to 40, 41 to 50, 51 to 60 . . . 91 to 100.
4. Counting 11 to 20.

   A helpful chart for these four steps in counting can be made by hanging 10 cardboard strips from the blackboard, each strip lettered as shown on page 152.
5. Counting by 100's to 500 and then to 1000; e.g., 100, 200, 300, 400, 500.
6. Counting 100 to 200 by 10's; e.g., 100, 110, 120, 130, 150, 140, etc.
7. Counting to 300, 400, 500, 1000 by 10's; e.g., 200, 210, 220, etc.

A second series of graded steps in counting gives exercises more closely related to work in addition than pure counting. These forms of counting include, counting by two's, three's, four's, etc., beginning with zero or a multiple of the number; counting backwards; and counting by any number starting at any base. The teacher is, therefore, referred to the topic Addition through Counting, in the following chapter, where both the method of presentation and the gradation are discussed.

What Should Counting Give the Child?—As the pupils progress through the various forms and graded steps in counting, their knowledge of number and number relations increases as a result of these exercises. Briefly summed up, counting gives the pupil: (1) The number series in proper order; (2) the number names; (3) the decimal rhythm of the Arabic notation; (4) the place of any number in the number series; (5) the basic meaning of number; (6) the basis for the fundamental operations; counting by two's is addition, counting backwards is subtraction, counting by two's paves the way for multiplication and inversely for division; (7) the primary facts of the tables.

Teaching Children to Write Figures.—*The Figures* 1–9. —When numbers are mastered through counting and measuring experiences, figures must be taught. No formal lesson should be given in reading figures through ten. Children learn to read figures incidentally, in the course of all lessons. The writing of the figures 1–9 must be taught as an arbitrary process by a deductive, imitative method with repetitions that make a multiple sense appeal. To teach the figure 7 the following steps may help:

1. Teacher writes a series of 7's on blackboard and explains the two characteristic lines in 7, viz., ¯ and /.

2. The children now imitate by making large 7's in the air. Large free-arm movement to impress direction and slope must be encouraged.

3. Visualize the 7. Children close eyes and attempt to " see " the 7, writing it in the air or with finger on the desk as the 7 is visualized.

4. Children now pretend to write 7 on an imaginary paper. The teacher corrects and keeps urging smaller forms until the deirable size is approximated.

5. Crayon and paper are now used and the 7 is attempted. Those who produce incorrect figures are sent to the board to trace the 7's there; the others are corrected

and encouraged to repeat the writing of the 7 until habit is formed.

Figures of Two Orders.—Patience is the only requisite for teaching children to write any digit. The difficulty in figure writing is not experienced until 11, 12, 14, etc., must be written. If the symbol ⅋ represented eleven, ϕ represented twelve, and ¶ thirteen, there would be no difficulty, because eleven, twelve and thirteen are new ideas and are represented by new symbols, ⅋, ϕ and ¶. But when **11** is eleven, **12** is twelve and **13** is thirteen, we have old symbols representing new ideas. In attempting to establish, for the child, an association between new number ideas and new arrangement of known symbols, we find the key to the difficulty experienced by pupils. To meet the pedagogical problem, teachers devised the plan of converting 13, 14, or 19 sticks into 1 bundle of 10 sticks, and 3, or 4, or 9 sticks over. The same was done with beads. Children given 13 beads put ten into a small bag and said, "Thirteen is 10 and 3 over." They were then taught to write 13, 14, 15, 16 beads, thus,

Bag	Beads
1	3
1	4
1	5
1	6

and later in closer proximity, thus, 13, 14, 15, 16.

While this recourse to sticks, beads, etc., is the only mode open to a teacher in foreign schools, it is needlessly artificial for American children who know the dime and the cents. The first step consists in giving oral drills, changing cents into dimes and cents; 15, 16, 18, 12, 14 cents are changed into 1 dime and 5, 6, 8, 2 or 4 cents respectively. The children then write dimes in one column and cents in another. A small silver circle heads the dime column on

the blackboard and a copper colored circle the cent column, and 15 or 16 is written as 1 under "dimes" and 5 or 6 under "cents." Another device is to make the 1 very large, since it is worth so much, hence the children write 1_5, 1_6, etc. But this leads to a habit of writing numbers which must soon be dropped. For a short time numbers between ten and nineteen are reduced to dimes and cents and are written as such. But as the days pass the signs of the ten's and the unit's place are eliminated and the numbers are not continued as dimes and cents. The child acquires the habit of writing 15, 17, 14, regardless of the objects referred to. Numbers of three orders may be taught by using dollar, dime and cent, equivalents. Drill must be given in writing numbers that are dictated, in reading numbers from arithmetic books or charts, and in writing, in figures, numbers that are spelled out in the textbook or on a chart. The aim must be to develop accuracy and instantaneous recognition or determination of numbers.

Roman Numbers.—*The Early Work is Incidental.*—The Roman notation must be taught in the elementary course for practical, as well as for cultural reasons, but no formal program need be arranged for this literal system. Roman numbers from I through XII may be introduced, incidentally, in teaching children to tell time, while giving drill lessons, or in the course of a game. In the early lessons on telling time, the clock used naturally shows figures in the Arabic form. When the child has made progress in time telling, a clock dial having Roman figures is introduced.

The children continue telling time with increasing accuracy; some do not even notice the change in the figure form. Teachers usually employ a circle containing the figures from 1 through 12 in giving drills in rapid manipulations. It is well to change the figures in the circle from Arabic to Roman form without explaining the change. Since children soon associate each figure with a fixed place

on the clock dial or in the circle, they give these new figures the same name as the old. So, too, in chalking boxes, indicating places, rows, etc., in the course of games, the Roman notation is used without explanation. The children learn these forms incidentally through natural use.

Later Work in Roman Notation.—In the primary grades, Roman numbers beyond XII need not be taught. Early in the grammar grades the Roman numbers that are known should be analyzed and through comparisons the following laws should be elicited from the children and definitely formulated:

1. Repeating a symbol repeats the value; e.g., one capital "I" is 1, but two capital "I's" represent 2 and three capital "I's" represent 3.

2. A smaller number before a larger number indicates subtraction; e.g., I before V means 5−1, hence 4; I before X means 10−1, hence 9.

3. A small number following a larger number indicates addition; e.g., V followed by I or II or III means 5+1 or 2 or 3, hence 6, 7, or 8; so, too, X and I or II represent 11 or 12.

The rules derived from an analysis of the Roman numbers I to XII, when applied, will lead children to write successive numbers. All Roman numbers are read and written, at first, by analytical and synthetical processes; through varied drills given regularly, for short periods, this recognition and motor expression gradually become habituated. Supervisors must grade this branch of notation and hold each succeeding class responsible for a definite series of new numbers. When Roman notation is introduced, incidentally, and assigned to a number of grades, children learn these new number forms without experiencing the burden of a task at any point in the school course.

CHAPTER VII

THE FUNDAMENTAL OPERATIONS

ADDITION AND SUBTRACTION

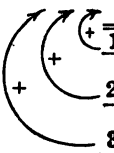
In the discussion of methods of teaching any branch of arithmetic, the reader will not find every step in the process detailed nor every topic graded and subjected to pedagogical analysis. Aside from considerations of limited space, this book aims to give only the general procedure, to indicate the high lights in method, in the hope that the teacher will apply the viewpoint and the essentials of methodology here indicated to the specific problems of instruction in arithmetic. Without the teacher's reaction, methods of teaching become empty forms, the imitation of which can lead to no useful purposes.

ADDITION

Incidental Addition.—The child who has learned his numbers through nine or twelve by a measurement process as well as through counting has added without a formal introduction of this process. Such a child would know, as a result of the lessons on "six," the combinations of 3+3, 5+1, 4+2, 2+2+2, 2+4, 5+1, 5+?=6, ?+2+1=6, 2+2+?=6, 3+?=6, etc. This basis of number facts gives a stock of useful addition combinations.

The Forty-five Combinations.—*What Are They?*—The addition of any two digits will give a sum no greater than eighteen. Only forty-five combinations can be made by adding any two digits, counting 8+9, and 9+8, as one combination. These combinations, when mastered, form the backbone of addition, for they are the basic habits with-

out which accuracy and speed in addition cannot be attained. A simple device for obtaining the forty-five combinations is found in the following diagram:



	1	2	3	4	5	6	7	8	9	
1	2	3	4	5	6	7	8	9	10	9 Combinations
2		4	5	6	7	8	9	10	11	8 "
3			6	7	8	9	10	11	12	7 "
4				8	9	10	11	12	13	6 "
5					10	11	12	13	14	5 "
6						12	13	14	15	4 "
7							14	15	16	3 "
8								16	17	2 "
9									18	1 Combination

45 Combinations

FIG. 8.

To obtain the first set of combinations, add 1 of the vertical column to each of the numbers in the horizontal line; this gives $1+1=2$, $1+2=3$, $1+3=4$, $1+4=5$, . . . $1+9=10$; to obtain the second group of combinations add 2 of the vertical column to each number in the horizontal line, thus, $2+1=3$, but since $1+2$ is found in the preceding line, this is omitted; $2+2=4$, $2+3=5$, . . . $2+9=11$.

Gradation of the Forty-five Combinations.—It is obviously important to arrange these forty-five combinations so that mastery is developed first in the simplest associations, and then to continue by graded steps to the most difficult. This group of additions was tested under experimental conditions by Courtis and arranged, on the basis of the evidence obtained, under five heads. The results of the investigation are given below. Any two combinations in

the same group are of approximately equal difficulty for the pupil.

Group I. Very Easy 0 0 0 0 0 0 0 0 0 0
0 1 2 3 4 5 6 7 8 9 1 2
and the reverse 0, 0, etc.

Group II. Easy 1 1 1 1 1 1 1 1 2 2 3 4 5
2 3 4 5 6 7 8 9 3 2 3 4 5
and the reverse,

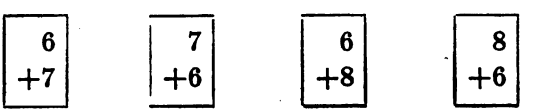
Group III. Average 2 2 2 2 2 3 3 3 3 3 4 2
3 4 5 6 7 2 4 5 6 7 5 7
and the reverse

Group IV. Hard 2 2 3 3 4 4 4 4 6 7 8 9
8 9 8 9 6 7 8 9 6 7 8 9
and the reverse.

Group V. Very Hard 5 5 5 5 6 6 6 7 7 8
6 7 8 9 7 8 9 8 9 9
and the reverse.

How to Teach the Forty-five Combinations

1. Focalize on two or three new combinations; e.g., 6+7, 6+8.
2. Counting exercises of the type:
   1, 2, 3, 4, 5, **6**, 7, 8, 9, 10, 11, 12, **13**.
   1, 2, 3, 4, 5, 6, **7**, 8, 9, 10, 11, 12, **13**.
   The numbers to be associated should be said with emphasis as they are met in the number series.
3. Analyze the composite thirteen into 6 and 7 or 7 and 6 by using some illustrative material, dots, beads, etc.
4. Drill on these combinations, using flash cards or perception cards.



5. Variety of Drills,
   (*a*) Algorism form, $\begin{array}{r} 6 \\ +\ 7 \\ \hline 13 \end{array}$
   (*b*) Equation form, 6+7 = ? 7+6 = ? etc.

(*c*) Decomposition form, $6+?=13$, $13=7+?$

(*d*) Drill by devices, using circles, clock games, etc.

6. Practical application of these combinations in real problems.

Objects to be Attained in Teaching the Forty-five Combinations.—Each day brings its repetition of combinations taught. As progress is noted a new combination or two is taken from the graded list and is taught by the plan outlined in the preceding paragraph. In all these drills two vital aims must be attained, *permanent fixation* and *instantaneous recognition.* It has been found experimentally that adding that is too slow can lead to no permanent and worthwhile results, because attention is permitted to wander, and lacks the intensity necessary for effective impression. Counting, as an aid in addition, should be discouraged after the initial step, so that the counting habit may not later arise to hinder column addition. Instantaneous recognition of an association reduced to habit must be the governing aim in all these early addition exercises.

Addition through Counting Exercises.—*Counting Exercises Graded.*—Much effective drill, in addition, can be given through a variety of counting exercises that involve a marked additive process. The steps in any of the forms of counting can be graded as follows:

1. Counting by 2's, 3's, 4's, etc., starting at 0.
2. Counting by 2's, 3's, 4's, etc., starting at a multiple of 2, 3, 4, etc.
3. Counting by 2's, 3's, 4's, etc., starting at a multiple of 2, 3, 4, etc., and going down.
4. Counting by 2's, 3's, 4's, etc., starting at 1.
5. Counting by 2's, 3's, 4's, etc., starting at any number.
6. Counting by 2's, 3's, 4's, etc., starting at any number, going down.

Steps No. 3 and No. 6 pave the way for drills in subtraction. The usual sequence of counting forms is as follows: 10's, 2's, 5's, 3's, 4's, 6's, 8's, 9's and 7's; experience rather than experiment indicates this order. Scientific study may prove either that changes must be made in the sequence or

that gradation in this respect is not as important a factor as most of us believe to-day.

Method of Teaching Additive Counting.—To teach children to count by 2 or 3, starting at 0, the following sense appeal may be employed:

(*a*) Count 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, etc.

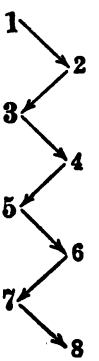
(*b*) Count 1, **2**, 3, **4**, 5, **6**, 7, **8**, 9, **10**, 11, **12**, 13, **14**, etc.
Teacher writes these two lines on the board. The children read the second line, emphasizing the even numbers, thus 1 (soft), **2** (loud), 3 (soft), **4** (loud), 5 (soft), **6** (loud).
Repeat a number of times until 2, 4, 6, 8, etc., have made their impress visually and orally.

(*c*) Count **2, 4, 6, 8, 10, 12**, erasing odd numbers. At first, taps, or arm or finger movements may accompany the count; later these are omitted. Experiments made in these counting forms show that the introduction of rhythmic quality and auxiliary movement of finger, arm, or foot will aid in the acquisition of the new series.

(*d*) Similarly (1) 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, etc.

(2) **1**, 2, **3**, 4, **5**, 6, **7**, 8, **9**, 10,
loud soft loud soft loud soft loud soft loud soft
11, 12, **13**,
loud soft loud

Other arrangements are



1	2
3	4
5	6
7	8
9	10
11	12 etc.

(3) **1 3 5 7 9 11 13** etc.

The mode of teaching counting by 2's applies to all other forms, i.e., by 3's, 4's, etc. As the number becomes greater, the linear sequence used above may not be convenient. Charts showing an arrangement in ten columns of ten numbers each are helpful; the following is a sample chart for counting by 4's:

1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	15	25	35	45	55	65	75	85	95
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	20	30	40	50	60	70	80	90	100

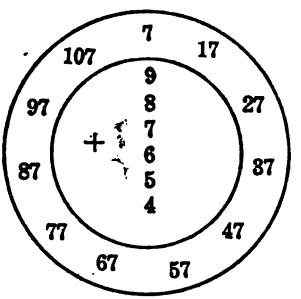
FIG. 9.

Many interesting inferences may be drawn from such a tabulation, depending upon the ability and the advancement of the children. The counting drills not only give practice in rapid addition, but also pave the way for mastery of

the multiplication tables, for the successive sums are endings of these tables and induce ratio ideas.

Carrying the Forty-five Combinations to Higher Orders. —The forty-five combinations cannot determine efficiency in real addition unless each one of the combinations is carried beyond the confines of twenty. Children recognize instantly 6+7, but hesitate for no little time when 26 and 7 or 46 and 7 are to be added. It is necessary, therefore, to give drills like the following if the forty-five combinations are to be made the means of developing rapid and accurate addition of long columns.

7	17	37	67	57	27	87	97
+6	6	6	6	6	6	6	6
6	36	16	66	27	86	56	96
+7	7	7	7	7	7	7	7



Add numbers in inner circle to the numbers in outer circle as rapidly as the teacher points to them; e.g.,

7+5, 7+55, 7+85, etc., 9+5, 9+15, 9+25, 9+45, etc.

The numbers in the outer circle may be erased and 6, 16, 26, 36, or 9, 19, 29, 39, etc., substituted.

Adults often find themselves hesitating in column addition when they reach a number like 59 and must add 8; they usually say to themselves or think, "59 and 8 sixty, . . . sixty, . . . 67." Few, if any, hesitate on " sixty " but may spend time on the new unit's figure. Nevertheless an analysis reveals the fact that these people can give the sum of

9 and 8 instantly. Evidently their early training in addition failed in a vital respect in that it did not seek to carry the associations of the forty-five combinations to higher orders so that a 9 and an 8 will, by force of habit, suggest a 7 in the answer. Two forms of drill must therefore be given:

(*a*) 2+7, 42+7, 52+7, 57+2, 87+2, 92+7.

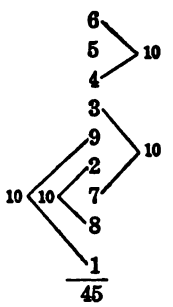
(*b*) 7+9, 27+9, 37+9, 59+7, 107+9, 109+7, etc.

Miscellaneous Drills in Oral Addition.—As soon as a sufficient number of habits in additive combinations have been developed, any form of addition should be used in drill exercises. The suggestions of the preceding chapter, viz., the circle, the clock, the triangle, the ladder, the railroad ties, column addition, etc., have their application here. Each of these mechanical devices should be used, so that children will not tire of any one of them. By these devices practice can be given in such additions, i.e., 9+7+8+5+6+4+9, etc., or in simple combinations, e.g., 5+9, 6+8, 7+6, etc. To these we may add oral addition of statistics in the appendix of the geography or the history textbooks; of columns of figures in arithmetic textbooks, in newspapers, or from any convenient source. Practice may be given, though not so frequently, in horizontal addition necessary in finding the totals as in the following tabulation:

Record of Parcels Shipped by Brown & Co. Week Ending...., 1916

Sent by	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total for the Week
Parcel Post..	72	148	106	203	117	48	?
Express.....	38	96	92	85	116	71	?
Freight.....	17	32	41	54	74	39	?
Messenger...	9	7	14	12	8	27	?
Daily Totals.	?	?	?	?	?	?	?

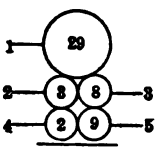
Aids in Rapid Addition.—Many devices have been developed for increasing speed in addition. The most helpful aid in developing rapidity is the acquisition of habits that promote accuracy. For this reason the device known as "catching ten's" is more effective in textbooks than in actual practice. Speed may be attained, but this practice makes for inaccuracy. Too often the child spends more time looking for the ten's than is required to add without this "rapid" device. The accompanying example in addition illustrates the process of "catching tens" required in many textbooks. A cursory glance shows that unless the child exercises unusual care some numbers will be lost in the selecting and combining processes, or that much time is spent looking for the numbers that combine to form 10. That addition is most rapid which is most accurate and requires a minimum of mental effort.



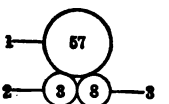
Rapid addition of numbers of two orders may be taught to good advantage. In the addition of 35 and 48, two processes can be used: A, 30+40, 5+8, 70+13=83; B, 35+40=75, 75+8=83. In form A there is danger that the child will lose the 70 (resulting from 30+40), by the time he has 13 (resulting from 5+8). The second form is commended because it is a continuous process and the child can be taught to think "35, 75, 83." Addition of three numbers of two orders each can be performed in the same way; e.g., $\begin{array}{r} 54 \\ 28, \\ 19 \\ \hline \end{array}$ 54+20=74, 74+8=82, 82+10=92, 92+9=101; the child thinks: "54, 74, 82, 92, 101."

This form of rapid addition must be graded before it is taught. Step 1 gives drill in adding 20 and 20, 40 and 50,

30 and 60, etc. Step 2 practices 24+20, 36+30, 48+50, 53+40, etc. Step 3 includes numbers like 24+35, in which the two numbers in unit's place give a sum less than ten. Step 4 introduces the forms mentioned above, viz., 48+56, 39+47, etc. To appeal to the visual minded child the sequence of the steps may be shown on the board in some graphic way with the aid of colored chalk, thus:



29, 59, 67, 87, 96



57, 87, 95

This form of oral addition is rapid and exceedingly useful in business activities.

Written Work in Addition.—*Delay Written Addition.*—Nothing is gained by an early insistence on written addition. Children should develop mastery in the forty-five combinations, in irregular counting, in carrying the forty-five combinations to higher orders, in adding 26 and 8, 9 and 34 before long single-column addition is attempted. In the final analysis, written addition involves no new mathematical operation and depends for accuracy and speed on efficiency in oral addition.

Column Addition.—Single column addition is only oral addition with the answer not spoken but written. Care must be taken to see that children do not acquire the habit of excessive speech in addition. In finding the sum of the single column, children should be taught to think, "9, 16, 21, 29, 35, 38 . . ." not "9 and 7, 16; 16 and 5, 21; 21 and 8, 29; 29 and 6, 35; 35 and 3, 38." Teachers in upper grades who ask children "to say aloud what you think when you add" find that a large

9
7
5
8
6
3

proportion of the class uses the second form and thus indulges in more language than addition.

54
73
89
97
21
83
2
7

In two-column addition, the new problem lies in "carrying" to the next higher order. In teaching children to add the accompanying numbers of two orders, review the name of each place, units and tens or cents and dimes; when the sum of the units, 27, is obtained let children analyze it into 7 cents and 2 dimes; the simplest process is to put *cents* in *cents column* and *dimes* in *dimes column* as is shown. Objection is made, by many, to the use of the auxiliary figure, 2, placed in the *tens column*. But it must be urged that the object in addition is to make carrying a habit in order to insure accuracy and speed. To keep the 2 in mind requires more effort, more time, and involves greater chances of error than to put it down in its appropriate column. Professional bookkeepers always put these "carrying figures" down at the foot of the ledger to keep as a check when they add the columns in the reverse direction.

In teaching addition of numbers of three orders, the same process may be followed, viz., a minimum of rationalization with algorism that lends itself most easily to habituation. Some texts give themselves to detailed rationalization which cannot appeal to the immature child and often hinders rather than helps in acquiring habits in addition. In adding 234 and 325, the following analysis is suggested by these books:

$$\begin{aligned} 234 &= 200+30+4 \\ +325 &= 300+20+5 \\ \hline 500+50+9 &= 500 \\ &\quad +\ 50 \\ &\quad +\ 9 \\ \hline &\quad 559 \end{aligned}$$

or

H.	T.	U.		
2	+ 3	+ 4		
3	+ 2	+ 5		
5	+ 5	+ 9	= 5H	= 500
			5T	= 50
			9U	= 9
				559

Experience shows that children do not demand any justification for most of the mechanical processes in arithmetic. To teach them the *modus operandi* first and evolve the rationalization when they are older and interested in the thought aspect of previous experiences will give most satisfactory results.

Gradation of Written Addition.—To guard against undue teaching difficulties in written addition, successive processes and steps should be arranged in a graded series, as indicated below:

A. No Carrying

1. Single columns—like oral work.
2. Two orders—

	14
	33
	42
	89

3. Three orders—

	352
	213
	112
	121
	798

B. With Carrying

1. (*a*) 8+7;
   (*b*) 18+7;
   (*c*) 28+7; like oral.
2. Carrying in 1 column, e.g., 18+7+7+6.
3. Carrying in 2 columns, e.g., 18+27+36+57.
4. Carrying in 3 columns.
5. Any group of figures.

With sufficient drill—a daily three-minute practice kept up through the grades—any normal child can attain the degree of speed and accuracy set by a reasonable standard.

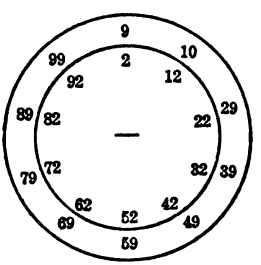
SUBTRACTION

Relation of Subtraction to Addition.—Knowledge acquired in one set of tables contributes to mastery in related processes. The child who knows his addition combinations finds subtraction tables much easier to acquire. So, too, knowledge of the multiplication tables paves the way for mastery of associations in division. Subtraction should, therefore, be taught almost synchronously with the addition process. The forty-five additive combinations afford an effective basis for teaching subtraction. The thorough mastery of 9+6 involves drills not only in 9+6, 6+9, but also 9+?=15, ?+9=15, ?+6=15, and later in 15−9=?, 15−6=?. The child, therefore, acquires the basis of subtraction, incidentally, while he learns numbers and the basic combinations in addition. In using the forty-five combinations for subtraction, we may, therefore, follow the same steps, devices, gradation, and mode of presentation. Care should be taken to use each combination in as many forms as possible. To illustrate: Drill on 16−7 must be given as 7+9=?, 9+7=?, 7+?=16, ?+7=16, 9+?=16, ?+9=16, 16−7=?, 16−9=?, 16−?=7, 16−?=9, $\begin{array}{r} 16 \\ -7, \\ \hline ? \end{array}$ $\begin{array}{r} 16 \\ -9, \\ \hline ? \end{array}$ 16 is how much greater than 9?, than 7?, $\begin{array}{r} ? \\ +7, \\ \hline 16 \end{array}$ $\begin{array}{r} ? \\ +9, \\ \hline 16 \end{array}$ 9 is how much less than 16?, 7 is how much less than 16? The teacher must guard against stereotyped forms in these mechanical drills.

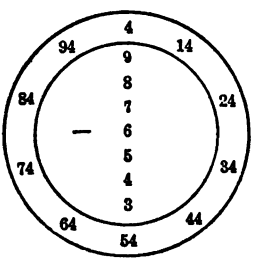
The Three Basic Ideas in Subtraction.—An analysis of the variety of forms possible for 16−7, as outlined above, will readily show that children must learn three basic con-

stituent ideas in the subtraction concept: The first is the "take away idea," 9 less 3 is what? 3 from 9, or 9, take away 3, leaves what?; the second is the "adding idea," what number added to 3 gives 9; the third is the "difference idea," 9 is how many more than 3? These three ideas of the subtraction concept must give form and purpose to the variety of drills suggested for the subtraction tables.

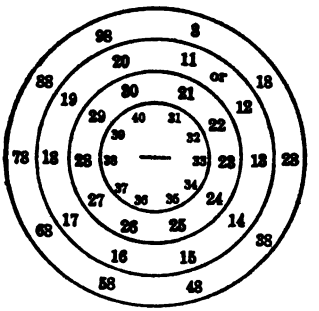
Carry the Forty-five Combination to Higher Decades in Subtraction.—A common weakness in addition performed by many adults, as well as children, is repeated in subtraction. Many who can give an instantaneous response to 17−8, hesitate when the subtraction is 57−8; these people usually think or actually say, "57 minus 8, −forty . . . forty . . . forty-nine," but when they see 17−8, they show no such hesitation. The reason is obvious—no teacher made the transition from the forty-five combination, to higher decades. Drill which will attain this end is absolutely essential; for this purpose, the following devices are suggested:



Subtraction in the same decade. 9−2, 19−12, 69−62, 89−82, 99−92.



Subtraction of a digit from any decade. 9 from 14, 24, 64, 54, 94. 5 from 14, 94, 84, 74.



Subtraction of any number of two orders from any other number of two orders. 38−19, 68−56, etc. Process: 38−19, 38−10, 28−9, 19. Reverse of similar addition device. See page 167 for procedure and gradation.

The numbers on the larger dial must be changed in each case until every variety of combination is introduced.

Miscellaneous Drills in Subtraction for All Grades.— The variety of drills that is suggested for addition (pages 163–5) can usually be applied to subtraction. To these, the following are added as supplementary:

(1) 1 2 3 4 5 6 7 8 9
−8

Teacher points to 9 and −8 or to 8 and −8, child gives 1 or 0. Teacher points to 7 and −8; when child says, " can't take 8 from 7 " the teacher gives permission to make any number ten higher, i.e., 6 becomes 16, 7 becomes 17, 4 becomes 14. Teacher points to 3 and −8, to 5 and −8, to 4 and −8, to 6 and −8 and the child thinks " 13, 15, 14, 16 " in each case and gives the answers, " 5, 7, 6, 8 " respectively. The subtrahend 8 is changed to any other number and the drill is repeated until the child gives the answers from force of habit.

(2) $52 - 27 = ?$ $52 - ? = 27$ $? - 27 = 25$ $27 + ? = 52$ $? + 27 = 52$, 52 is how much greater than 27? 27 is how much less than 52? Required process: *A*. 52−27; 52−20, then take 7 from the result; e.g., child thinks 52, 32, 25. *B*. Addition Idea—what number added to 7 gives two, etc., as will be explained under Austrian Subtraction.

(3) Present

52−27
92−60
74−26

	a	*b*	*c*
	52 −	27 =	25
	92 −	60 =	32
	74 −	26 =	48
	218 −	113 =	105

Call first column *a*, second *b*, the answers *c*; then add all of *a*, all of *b*, all of *c*; take all of *b* from all of *a* to find if total *c* results.

218−113, 218−100=118
118− 10=108
108− 3=105

(4) Had 326; lost 15, 33, 54; left?
There are three possible forms of solution:

A

326—total lost. Add total lost;
15+33+54=102
326−102=224

B

326 minus successive losses

326
− 15
311
− 33
278
− 54
224

C

Perform as continuous addition

362
15
33
54
224

Think 4, 7, 12; 2 and ?=6, hence 4. Carry 1, think 6, 9, 10; 0 and ?=2, hence 2. Carry 1, think 1 and ?=3, hence 2 in answer.

It is obvious that Form I requires two mathematical and two physical processes. Form II, one mathematical and three physical processes; Form III, one physical and one mathematical process. To pupils in the higher grades, e.g., fifth or sixth years, mechanical processes like subtraction are made interesting through such arrangements.

Written Subtraction.—*No Carrying or Borrowing.*—When mastery of subtraction combinations is attained, written work may be introduced. The first type involves no borrowing because every order in the subtrahend is smaller than the corresponding order in the minuend, e.g., 89, 57,
-24 -35

etc. This type teaches children the algorism required in all subtraction work. The mode of solution and explanation will be given in the more difficult forms of subtraction.

The Borrowing or Carrying Idea Introduced.—There is obviously no difficulty in the first type of written work. Difficulty in written subtraction is introduced when a place in the subtrahend is larger than the corresponding place in the minuend, e.g., 72. This type of work is preceded by
-25

examples in which the unit place of the minuend is zero and the unit place of the subtrahend is larger, e.g., 70, 60,
-24 -36

etc. The mode of presentation and rationalization is the same as in the case of examples like $72-58$ and is given in the study of this type.

There is a large variety of solutions for the subtraction of numbers typified by $72-58$, $93-68$, etc. Let the reader perform such subtraction and endeavor to make conscious and rational each step in the solution. With this process clearly fixed in mind, the succeeding discussion may gain in significance. We shall sum up briefly, in tabular form, common modes of teaching subtraction.

1. "Borrow and Pay Back."

72
−58
14

Child says: 8 from 2 I can't take, I borrow a ten from the 7 tens and now I have 12; 8 from 12 is 4. Now I pay back 1 ten to 5 tens, since I borrowed from the tens before; 6 from 7 leaves 1. Ans. 14.

Child thinks: 8 from 12, 4; 6 from 7, 1.

Criticism: 1. Needless terminology.

2. Does not rationalize. Child does not understand why we pay back to the lower ten having borrowed from the upper ten.

3. Takes longer to become habit.

4. "Borrow and pay back," is not a mathematical conception.

2. Method of Equal Additions.

72
−58
14

Child says: 8 from 2 I can't take. I add 10 units to the 2 units and 1 ten to the 5 tens and do not change the value, hence units become 12 and tens 6. 8 from 12, 4; 6 from 7, 1. Ans. 14.

Child thinks: 8 from 12, 4; 6 from 7, 1.

72−58=
82−68,
hence
10
72
1
58

Criticism: The form is more mathematical and the cumbersome terminology is gone, but all the other limitations remain. This method is the mathematical explanation for the former one. While it is axiomatic to the adult mind that "if to equals we add equals, the result will be equal," the child does not comprehend this.

In examples like 3203−1654 the auxiliary figures and the equal additions lead to confusion.

3. Complement of Tens.

73
−58
15

Child says: 8 from 3 I can't take; 8 from 10 is 2; now I add the 3 and I obtain 5. Remainder same as previous methods.

243
−158
85

Child thinks: 10−8+3=5; 5+1=6; 7−6=1.

Criticism: The method is old and causes no little trouble before subtraction is reduced to a habit. In the second example the child would say: 10−8+3=5; 5+1=6; 10−6+4=8; 1+1=2, 2−2=0. Ans. 85. The possibilities for confusion are clear.

4. Method of Decomposition or Reduction to Equivalents or Borrow Method.

6 10
~~7~~2
−58
14

Child says at first: 8 from 10 I can't take; I change 1 ten into units; this leaves 6 tens and gives 12 units; 8 from 12, 4; 5 from 6, 1. Ans. 14.

(Auxiliary figures here used to make simpler the explanation. They should not be permitted with the average child.

72
−58
14

Child acquires habit of thinking: 8 from 12, 4; 5 from 6, 1. Ans. 14.

10
1 3 10
~~2~~~~4~~3
−158
85

Criticism: (*a*) Simple. Can be shown concretely by use of dimes and cents.

(*b*) Experience shows it is readily mechanized.

(*c*) It is an explanation that children can comprehend. The term "borrow" is a misnomer, but the method is often referred to by that name.

−243
−158
85

(*d*) Arguments against it are: (1) it requires the mastery of separate subtraction combinations, and (2) when two ciphers occur in the minuend, 3004−1768, the child is confused by the successive reduction to equivalents, e.g.,

				Form recommended
	9	9		
2	~~1~~0	~~1~~0	10	
~~3~~	~~0~~	~~0~~	4	3004
−1	7	6	8	1768
1	2	3	6	1236

It may be argued that subtraction forms of the tables are necessary in all methods and the case cited is difficult by any process.

5. SUBTRACTION BY ADDITION: THE AUSTRIAN METHOD. Subtraction may be looked upon as a process of diminution and a process of comparison. A fuller meaning of the subtraction process may be obtained by analyzing the following examples:

A	*B*
John had 25¢ and lost 20¢. What had he left?	John had 25¢, William 20¢. John has how much more than William?
(1) The subtrahend was originally part of the minuend.	(1) The subtrahend a separate quantity, hence it was never part of the minuend.
(2) The result, 5¢, is a remainder.	(2) The result here is a measure of difference.
(3) This example requires 25¢ for objective presentation.	(3) This example requires 45¢ for objective presentation.
(4) The process: 25¢ − 20¢ = 5¢, hence one of diminution.	(4) The process: 20¢ + ?¢ = 25¢, hence one of comparison and difference.

It is obvious, therefore, that the additive idea or the process of comparison to find difference must be made an important part of the general subtraction process.

The Austrian method holds that subtraction can be reduced to addition and therefore taught not as a new process, but as an application of the older one. "John has 5¢. and William 7¢, how much does John need to have as much as William?" reduces itself, naturally, to John's money plus how much will equal William's, or 5 + ? = 7. If the child knows the forty-five combinations of addition, he knows the missing number, 2. Hence children are taught to subtract 52 from 87 as follows:

87
52
—
35

Child says at first: 2 and how many will give 7? Ans. 5; 5 and how many will give 8? Ans. 3.

Child thinks: 2 and **5**, 7; 5 and **3**, 8. Ans. 35.

After the child has mastered the type which requires no reduction or carrying, examples like 90−60, 94−30 are presented and the same process of addition is applied with no trouble. This form of example is followed by 116−45, in which there is no difficulty in the units place. The child solves this as follows:

116
45
—
71

Child says at first: 5 and ?=6, Ans. 1. 4 and ?=11, Ans. 7. Hence answer is 71.

Child thinks in later stage: 5 and **1**, 6; 4 and **7**, 11; Ans. 71.

In types, 72−58; 456−267, the vital element is introduced in the subtraction process. The mode of explanation and the final form of habitual solution are here offered.

72
58
1
—
14

Child says at first: What +8 gives a 2; Ans., a 4, 4+8=12; put down 4 in units place and carry the 1 to next column (as in addition). 5+1=6; 6+?=7, Ans. 1.

Child thinks when habit is developed: 8 and 4 =12; 5 and 1= 6, 6+1=7. (Auxiliary figures inserted to aid explanation. Not necessarily part of the solution.)

456
267
11
—
189

Child says at first: 7 and ? will give a 6. Ans., a 9; 9+7 is 16, put down the 9 and carry the 1. 6+1=7. What added to 7 will give a 5? Ans., an 8; 8+7=15, put down the 8 and carry the 1. 2+1=3; what added to 3 gives 4? Ans. 1.

Child learns to think: 7 and **9**, 16; 7 and **8**, 15; 3 and **1**, 4. Ans. 189. (See note above *in re* auxiliary figures.)

Gradation for Austrian Method of Subtraction.

Type 1.	4 +? 9	Introduced through comparison form of example.
Type 2.	58 −34	Merely an extension of the first type to numbers of two orders.
Type 3.	96 105 115 97 −30 −42 −60 −? 50	
Type 4.	90 90 −? 45 45	
Type 5.	321 175 146	

The gradation suggested is merely to introduce the successive difficulties in the Austrian method, not to cover every type in subtraction. Complete gradation in subtraction will be found in any good textbook in arithmetic covering the work of the earlier grades.

Auxiliary Figures in Subtraction.—Strenuous objection is made to the introduction of any auxiliary figures in the course of written subtraction. Teachers therefore force the children to omit the carrying figures seen in the accompanying work:

$$\begin{array}{r} 321 \\ 175 \\ {\scriptstyle 1\,1} \\ \hline 146 \end{array}$$

It was shown in a preceding discussion that a final decision concerning the use of auxiliary figures cannot be formulated so readily. Some individuals cannot visualize easily and

therefore cannot form the habit of carrying the figure in mind; they find themselves forced into adopting some cue to recall the carrying figure. The object in all this work is to develop accuracy and speed. Some individuals can attain a greater degree of accuracy and speed with auxiliary figures while others cannot. It is advisable to encourage children to work with as few crutches as possible, but the recalling device should not be forbidden for all types of children.

Estimate of the Austrian Method of Subtraction.—The sponsors of subtraction by addition argue that many advantages are derived from its use. First, no new combinations or processes need be learned. The tables, the mode of thought, the form of the association, the process of "carrying" in addition, are all utilized directly in this system of subtraction. Second, it is the method that experience teaches the small shopkeeper to adopt in making change. When the purchaser of a 69-cent article gives the merchant a dollar, the latter gives the former the amount necessary to add to 69 cents to equal a dollar, by counting "69 and 1 makes 70, and 5 makes 75 and 25 makes 1 dollar." Third, the method of getting ledger balance always seeks the number which, added to the smaller, will give the latter. Fourth, it is a mode of subtraction which proves the work while the subtraction is being performed. In any other method of subtraction, the mode of proof consists of adding the answer and the subtrahend to obtain the minuend. In the Austrian subtraction this addition goes on all the time so that when the subtraction is complete every item has been automatically checked.

But, teachers urge, the Austrian method of subtraction cannot be explained to children and requires more time for habituation; the pupils see no reason for the procedure, but learn it through mechanical repetition. In the method of reduction to equivalents, concrete demonstration is simple.

For this reason, it is argued, children do not acquire the subtraction habit so readily by this additive method. Another cause of the dissatisfaction expressed by teachers is that the "take away idea" cannot be eliminated from arithmetic; that the child may *say*, "What number *added* to 7 gives 2?" but the mind really subtracts. The adult who has always subtracted by a diminishing process finds the answer to the question $7+?=9$ by taking 7 from 9; but the child who has not yet been made the slave of a single mode of operation may obtain the "2" additively because the three numbers 7, 2, 9 are inseparably linked in the mind. Although the Austrian method is taught, the subtraction idea should not be eliminated altogether. The drills suggested in the earlier sections of this chapter include the form $9-7=?$ as well as $7+?=9$.

The controversy now raging over the method of subtracting by addition must be settled by scientific experiment and not by an exchange of arguments based on personal opinion. Classes must be taught under the same conditions by rival methods and then subjected to the same tests until sufficient evidence is gathered to warrant a conclusive verdict. Thus far, the Courtis tests applied in schools where the Austrian method is used show that this additive method is no more effective than some of the older, or even the discarded methods; that it does not give habits of greater speed and increased accuracy.

General Suggestions for Teaching Addition and Subtraction.—Throughout the work in addition and subtraction proof and checks should be required of the pupils. The relative worth of methods of proving and checking mechanical operations will be treated at the end of the discussion of the four fundamental processes.

The aim throughout this early work should be to mechanize the process as soon as possible. Explanations may be attempted with the abler pupils, but only comparatively

few are capable of comprehending the *rationale* of these operations. Seven- or eight-year-old pupils repeat explanations, but these statements are words, not ideas, to them.

In the attempt to habituate a process, teachers must not lose sight of the practical application. Children must not only be able to subtract 289 from 508 with accuracy and speed, but must also recognize the "subtraction idea" in a practical situation. There is always danger that the business applications will be sacrificed to the mechanical forms in drills in fundamental operations.

A final caution is not to emphasize difficult forms so that simpler relations are not recognized when they occur. Children who can divide 54.346 by 2.007 are often confused when asked to divide .1 by .01 or 2 by .2. So, too, children who can subtract 2465 from 5009 err when they are told to take 3564 from 8689, because they continue to "borrow and carry" from force of habit. The greater one's experience with business needs in arithmetic, the more is one impressed with the prevalence of the simple rather than the difficult forms in actual industry or commerce.

CHAPTER VIII

THE FUNDAMENTAL OPERATIONS (CONTINUED)

MULTIPLICATION AND DIVISION

Incidental Introduction of Multiplication.—Many courses of study in arithmetic set a definite grade, the second half of the second year or the first half of the third year, for the introduction of multiplication. We saw that the child learns 3 three's, 2 four's, 2 two's, etc., when number is taught through a measurement process. Although no formal tables are developed, the "multiplication idea" is, nevertheless, introduced and applied. A second informal preparation for multiplication is found in the drills in counting by 2's, 3's, 4's, etc., beginning at 0 or at a multiple of the number. The child who counts 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36 is learning the endings of the multiplication tables. It must be recognized, therefore, that when multiplication is formally introduced, valuable preparation has already been given.

The Multiplication Tables.—The final efficiency in multiplication is determined by the habits developed in the tables. We must stop, therefore, to consider important suggestions governing the teaching of the multiplication tables.

1. *Grade the Multiplication Tables.*—Not all multiplication tables are equally difficult for children. Theoretically 2×7 should be as difficult for a child as 8×7, but practical experience refutes this speculative hypothesis. Teachers give evidence that the tables of two's, four's, five's and tens are "easy," that the three's and nine's come next in

difficulty, and that the six's, seven's and eight's are in the third group of this gradation. The tables of eleven's and twelve's need not be considered seriously because all their items except 11×11, 11×12, and 12×12 are found in the earlier tables. In teaching the tables the most effective sequence seems to be the following:

Group I, 2, 4, 5, 10 times tables.

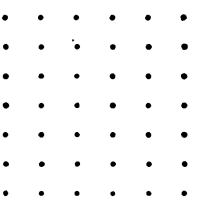
Group II, 3 and 9 times tables.

				Comment
Group III,	6× 2 6× 3 6× 4 6× 5	7× 2 7× 3 7× 4 7× 5	8× 2 8× 3 8× 4 8× 5	These combinations were studied in the tables of Groups I and II.
	6× 6 6× 7 6× 8	**7× 6** 7× 7 7× 8	**8× 6** **8× 7** 8× 8	Items in black figures are not difficult. The others are difficult combinations.
	6× 9	7× 9	8× 9	These items were studied in the 9 tables.
	6×10	7×10	8×10	These items were studied in the 10 tables.
	6×11	7×11	8×11	11× means repeat the figure, e.g., 66, 77, 88, etc., hence not difficult.
	6×12	7×12	8×12	Difficult.

Group IV, 11 and 12 times tables. All combinations except 11×11, 11×12, and 12×12 were studied in Groups I, II and III.

2. *Tables Evolved by the Pupils.*—Every combination and every product should be discovered by the children.

To present a carefully prepared chart with a table worked out from the beginning, " ×1," through the end, " ×12," is both confusing and discouraging to the child. To develop the " 6× table " the teacher begins by explaining to the class the need for the new table and then writes the first combination, 6×1=—. The class offers obvious answer " 6." This is followed by 6×2, 6×3, 6×4, and 6×5, all of which come from previous tables. To find the product 6×6, the children may have recourse to adding a column of 6 sixes; to find 6×7, a grouping of dots may be used.



To find 6×8, a rectangle 6×8 is drawn with all its square inches marked. To find 6×9, a line two yards long is drawn on the board with the inches marked in various colors at important subdivisions, e.g., 4, 8, 10, 12, 15, 16, 20, 24, . . . 72. A string 6 inches long is laid off 9 times and the distance 54 inches is covered. By one concrete device or another, children find the product of each succeeding combination and watch the table grow.

3. *Utilize the Commutations.*—The law of commutation in arithmetic is summed up in the equation $a \times b = b \times a$ when both a and b are abstract. It is important that the child be taught 6×7 and 7×6, 6×8 and 8×6 in the same period. It might be supposed that knowing 6×8 would guarantee a knowledge of 8×6, but experience teaches the contrary. The association, *a*, *b*, *c*, *d*, *e*, is mastered in regular alphabetical form, but the sequence, *e*, *d*, *c* or *g*, *f*, *e*, comes with greater difficulty. So, too, knowing 6×7 does

not assure us of equal mastery of 7×6. To teach every combination with its commutation almost reduces the items in the multiplication table by half. These commutations can be discovered by the children by the devices suggested in the preceding paragraph.

4. *Show Useful Mnemonics.*—Every means of added association should be given to children learning multiplication. In the five times table it should be shown that the products end in either 0 or 5, 0 when the multiplier is even, 5 when it is odd. In the nine times table the digits of the product always add up to 9. Thus in the accompanying column, 1 and 8=9, 2 and 7=9, 3 and 6=9, 4 and 5=9, 5 and 4=9, etc.; the unit digits decrease by 1 from 9, to 8, to 7, to 0; the digits in the second column increase by 1 from 0, to 1, to 2, to 3, etc., to 9.

9
18
27
36
45
54
63
72
81
90

In the eleven times table the product merely repeats the multiplier; thus, 11×5=5, 5, i.e., 55; 11×7 =7, 7, i.e., 77. These mnemonics may serve to fix difficult combinations for some children.

5. *Give Variety of Forms to Each Combination.*—Multiplication tables may be made an effective preparation for division by translating 7×9 into its possible forms, e.g., 9×7=?, 9×?=63, ?×9=63, ?×7=63, 7×?=63, 63=7 ×?, 63=?×9; the division forms, 63÷7=?, 63÷9=?, etc., may follow soon after the multiplication forms. This variety not only paves the way for division, but also maintains interest and stirs thought in the mechanical combinations.

6. *Show Children How to Find a Combination.*—Pupils should be shown how to orientate themselves in the tables should they happen to have forgotten an item in one of them. Let us assume that "7×8" is the combination that has been forgotten. The product can readily be found by one of three devices suggested in arithmetic books:

(*a*) Count by 7 or 8, beginning at any known multiple of 7 or 8, thus, 7, 14, 21, 28, 35, 42, 49, **56** or 35, 42, 49, **56**, or 40, 48, **56**, etc.

(*b*) Add 7 or 8 to the product of the combination immediately preceding 7×8, if that is known, e.g., $49+7$ or $48+8$.

(*c*) Subtract 7 or 8 from the product of the combination immediately following 7×8, if that is known, e.g., $63-7=56$ or $64-8=56$.

The second and the third suggestions usually require too much computation to prove valuable; the first device is of decided worth in finding a product in one of the tables.

7. *Carry the Multiplication Tables to Higher Orders.*—As soon as the progress of the child will permit, the combinations in the tables should be carried to higher orders, the products being found by analogy, e.g., $7\times12=84$, $70\times12=840$, $7\times120=840$; $8\times9=72$, $8\times90=720$, $80\times9=720$. The child may be told arbitrarily to add the cipher and multiply as if the numbers had no cipher. Such drills, given occasionally, tend to widen the scope of the tables.

8. *Introduce the Ratio Idea of Number through the Tables.*—As mastery of the tables increases, an interesting application will be found by utilizing the combinations for ratio solutions of problems. Children who know the two and the three times tables can be taught to solve the following problems:

2 apples cost 6¢, 4 apples cost ?
4 oranges cost 12¢, 2 oranges cost ?
3 pears cost 5¢, 9 pears cost ?

In the introductory stage, this type of ratio problem is solved by using a simple diagrammatic presentation, but as the mode of thought and the process of solution become better known, all objective aid is eliminated. As each new table is learned, a new variety of ratios can be incorporated into these problems.

Drill in the Multiplication Tables.—There is no royal

road to mastery of multiplication tables. Regular and persistent drills are the price that must be paid. All these suggestions for developing the multiplication tables are designed to show the reason and the system in the combinations; they must be followed by daily drills that use all the devices enumerated in addition combinations, viz., the circle, the clock, the ladder, the staircase, the relay race, the railroad ties, etc. In addition to these, an arrangement of numbers like the following proves helpful:

	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	110	132	144

FIG. 10.

Here we have the count by 2's, 3's, 4's, etc.; the ending of all the tables; the product of any two numbers through

12×12. This shows the peculiar products in the 5, 10, 9 and 11 times tables. Such arrangement can be used to drill on factors and multiples; it serves to aid children in answering such questions as, "What two numbers when multiplied give 24 or 36?" Another tabulation can be made by writing the numbers 1 through 12 in the vertical and in the horizontal lines and leaving the rest blank; the children are then asked to write in the proper space, 24, 48, 36, 18, showing what two numbers will produce them. An interesting series of competitions and games can be devised through this tabulation.

How to Teach a Combination of the Multiplication Tables.—To teach any combination, e.g., 6×7, the following steps may be helpful:

1. Count by 6's and by 7's, emphasize 42 in the series.
2. Analyze the combination 6×7 by diagram, yardstick device, dots, area device, etc., as explained.
3. Elicit the commutation $6 \times 7 = 7 \times 6 = 42$.
4. Perception card or flash card drill

7 $\times 6$	6 $\times 7$

5. Variety of drills:

   (*a*) Algorism
   $$\begin{array}{r} 7 \\ \times 6 \\ \hline 42 \end{array}$$

   (*b*) Equation $6 \times 7 = ?$ $7 \times 6 = ?$

   (*c*) Elliptical form $6 \times ? = 42$. $? \times 7 = 42$.

6. Practical problems involving 6×7 and 7×6.

Written Multiplication.—As in the other fundamental operations, the teaching problems in written multiplication arise from lack of knowledge of basic combinations and poor gradation. Accuracy, speed, and proper algorism in written forms of multiplication are automatic results of regular

drills. Assuming mastery of multiplication tables, the essentials of the further work in multiplication may be summed up in the following conditions.

Multiplying Any Number by a Number of Digits.—1. THE GRADATION.—Six distinct steps of increasing difficulty must be recognized in this type of multiplication; these are tabulated with reasons for the gradation.

(*a*)	214 312 221 ×2 ×3 ×4	(*a*) No carrying. Each multiplication is complete.
(*b*)	614 523 912 ×2 ×3 ×4	(*b*) The last number, the 6, the 5, the 9, is so large that when multiplied it gives two places.
(*c*)	46 225 114 ×2 ×3 ×4	(*c*) The carrying is introduced in one number but the carrying figure is 1.
(*d*)	53 212 ×5 ×7	(*d*) Steps (*b*) and (*c*) are combined.
(*e*)	346 268 ×3 ×2	(*e*) Carrying in more than one place, $2\times8=16$, carry 1; $2\times6=12$, $12+1=13$, carry 1 again.
(*f*)	346 536 ×4 ×6	(*f*) Carrying one or more times but the carrying figure may be more than 1, e.g., $6\times6=36$, carry 3, etc.

2. THE RATIONALIZATION PRECEDING THE DRILL.—This type of multiplication may be explained either by analysis or by comparison to addition.

(*a*) By analysis:

$$\begin{array}{r} 214=200+\ 10+\ 4 \\ \times2\quad\times2\quad\times2\quad\times2 \\ \hline 400+\ 20+\ 8=428 \end{array}$$

Then the child is shown the direct or the usual mode of multiplying.

A second form of analysis is often used.

$$\begin{array}{rrrrl} & \text{H} & \text{T} & \text{U} & \\ 216= & 2+ & 1+ & 6 & \\ \times 2 & \times 2 & \times 2 & \times 2 & \left\{\begin{array}{r} 400 \\ 20 \\ 12 \\ \hline 432 \end{array}\right. \\ \hline & 4+ & 2+ & 12= & \end{array}$$

The direct process is shown to be a duplication of the longer process of analysis. As a rule, these forms of rationalization are found to be too long and incapable of making their appeal to the young child.

(*b*) Through addition:

$$\left.\begin{array}{r} 214 \\ \times 2 \end{array}\right\} = \begin{array}{r} 214 \\ +214 \\ \hline \end{array}$$

The class is led to add 4 and 4 and conclude 2 four's are 8; 1 and 1 are 2, therefore, 2 one's are 2; 2 and 2 are 4 but 2 two's are four. The teacher performs both forms on the board and leads the children to see that the first is the shorter and, therefore, the better. Similarly,

$$\left.\begin{array}{r} 216 \\ \times 3 \\ \hline 8 \end{array}\right\} = \begin{array}{r} 216 \\ 216 \\ 216 \\ {\scriptstyle 1} \\ \hline 8 \end{array}$$

Children are led to say, "6, 12, 18, but 3 six's are 18, put down 8 and carry the 1." (The teacher does so in each case.) "1, 1, 1, are 3 and 1 to carry is 4, but 3 one's are 3 and 1 to carry are 4." (Teacher performs the operation in both cases.) "2+2+2 are 6 but 3 two's are 6." (Teacher completes process in both cases.) This form of rationalization requires little time, presents the reason for the process and

the form in a way that young children understand, and leads to the conclusion that multiplication is only the short form of addition.

To Multiply Any Number by Two Orders.—1. THE GRADATION.—Four important steps must be recognized in this type of multiplication.

(*a*) To multiply by 10.	(*a*) This is necessary to explain step (*b*).
(*b*) To multiply by 14, 16, 18, 19.	(*b*) It combines step (*a*) with previous knowledge.
(*c*) To multiply by 20, 30, 50, 60, 40.	(*c*) It is the necessary transition to (*d*).
(*d*) To multiply by 23, 42, 54, etc.	(*d*) It combines the previous steps in one example.

2. THE RATIONALIZATION.—(*a*) To multiply by 10: Let the children examine the products in the 10 times table and compare them with the multiplicand. Then ask them to solve 23×10, 42×10, 57×10, multiplying by ten as if it were a number of one order. The products thus obtained are again compared with the multiplicands and the obvious conclusion is now drawn. The children have thus learned that the short form of multiplying by 10 is to add a zero to the multiplicand.

(*b*) To multiply by 16, 14, 13, etc.:

$$\begin{array}{r} 42 \\ \times 16 \\ \hline 252 \\ 420 \\ \hline 672 \end{array} \quad = \quad \begin{array}{r} 42 \\ \times 6 \\ \hline 252 \\ 420 \\ \hline 672 \end{array} \quad + \quad \begin{array}{r} 42 \\ \times 10 \\ \hline 420 \end{array}$$

Elicit that $42\times16=(42\times10)+(42\times6)$. Arrange the work as shown above, the second column containing no new element. The teacher then shows how much simpler it would be to multiply 42×6 in the original example and thus

obtain $\begin{array}{r} 42. \\ \times 6 \\ \hline 252 \end{array}$ But 42 must also be multiplied by 10. This is done and the new partial product is put under the first partial product, $\begin{array}{r} 42 \\ \times 16 \\ \hline 252 \\ 420 \end{array}$; the two partial products are now added as in the long process and the answer is obtained. For some time the zero is placed in the units place and the expression "skip a place," is carefully avoided. When the class is ready for steps (*c*) and (*d*), which are taught in this same way, the children leave the units place in the second partial product vacant.

To Multiply by Numbers of Three Places.—To teach multiplication by numbers of three orders the same form of gradation and rationalization is used.

243	243		243		243
×123	×3	+	×20	+	×100
729 ←	729		4860		24300
4860 ←	4860				
24300 ←	24300				
29889	29889				

Short Process shown to be merely a repetition of the longer one.

Long Process shown first.

In the initial stages, the ciphers may be introduced, but when the sequence of steps has been grasped by the child, the multiplication should proceed by using units, 3, then tens, a 2, then hundreds, a 1, as the multiplier, placing the second and the third partial products in the correct relative positions.

Further Gradation.—The further organization of the

successive types in multiplication can be obtained from any good elementary text. Care must be taken with the common varieties of multiplication that contain ciphers, e.g.,

204	270	386	380	3800	234	2004	2356	207
236	542	540	540	540	206	1236	2004	604, etc.

The exact and most scientific grading is not as important as introducing only one difficulty in each new type of multiplication. The method of teaching these various forms of increasing difficulty follows the general course already outlined. As in the case of the other fundamental operations, attention must be paid to the practical application of multiplication in bills, problems of areas and volumes, finding costs of quantities, and in the host of business situations. In addition to developing skill in multiplication, children must be keen in the recognition of a condition that necessitates multiplication.

DIVISION

Division the Test of Mastery of Fundamentals.—Division, though considered the most difficult of the fundamental operations, causes less strain to the child than other examples of equal length, because of the variety of the work involved. It combines multiplication and subtraction with division. The Courtis tests tend to prove that once children learn the process of division they do better work in it than in multiplication. Weakness in mastery of the multiplication and subtraction tables will show very readily in pupils' work in division.

The Tables in Division.—Careful and varied drill in the multiplication tables guarantees ready mastery of combination in division. The child who has been taught $5 \times 9 = ?$, $? \times 9 = 45$, $5 \times ? = 45$, $45 = 5 \times ?$ and $45 = ? \times 9$, can readily make the transition to $45 \div 9 = ?$, $45 \div 5 = ?$, $45 \div ? = 9$, $45 \div ? = 5$, $5)45$, $9)45$. The teaching steps, the gradation,

the mechanical devices, the variety of drill, the practical applications—all these, emphasized in the multiplication tables, must be repeated in the new form, division.

Introducing Written Division.—1. *Careful Gradation.*—The form or the algorism in written division must be introduced through problems which apply the facts of the table. The solution of the problem, "At 6¢ a book, how many can I buy for 54¢?" is shown as $6\overline{)54}$. This, strictly speaking, is
9
a problem for oral, not written, solution, but it may be used for transitional purposes. The main teaching difficulty in division resolves itself into a matter of gradation. In teaching the division of a number by a number consisting of a digit, the following gradation should be observed:

Step	*Reason*
(*a*) $2\overline{)64}$	(*a*) No "carrying" or reduction, no remainder.
(*b*) $2\overline{)65}$	(*b*) No "carrying" or reduction, but the difficulty of the remainder must be met.
(*c*) $2\overline{)74}$	(*c*) The "carrying" or reducing problem is here introduced, but there is no remainder to add to the difficulty.
(*d*) $2\overline{)75}$	(*d*) Both difficulties of steps (*b*) and (*c*) are here introduced.
(*e*) $2\overline{)726}$, $2\overline{)736}$, $2\overline{)735}$	(*e*) The difficulties of the above steps with numbers of three orders. Here there may be two processes of "carrying" and a remainder.
(*f*) $2\overline{)126}$ $2\overline{)136}$ $2\overline{)137}$	(*f*) The problem arising from the fact that the hundreds figure in the dividend is too small to contain the divisor. Previous difficulties then introduced.
(*g*) Same steps, divisors 3, 4, etc., 12.	(*g*) The gradation is here obvious.

2. *The Rationalization.*—The mode of explanation of the process is here shown in summary form.

(*a*) $2\overline{)64}$

Elicit the short process and show that it repeats the long form.

	3	2
	dimes	cents
2	6	4

or

	3	2
	tens	units
2	6	4

Long process shown first.

(*b*) $2\overline{)65}$

	3	2 and 1 cent over
	dimes	cents
2	6	5

The children are told to express the remainder in the form used above or in the form "remainder over divisor"; no reason is given. The two forms of writing the remainder should be taught in the course of the primary grades but not simultaneously.

(*c*) $2\overline{)74}$

	3	7
	dimes	cents
2	7	4

2 into 7 = 3 and 1 over. But the 1 is a dime. Change it to cents and add it to 4 cents.

2 into 14 is 7.

No rationalization should be given beyond that offered for the division by two. The process should be reduced to habit as readily as possible through repeated solutions of a large number of graded examples. No auxiliary figures should be allowed with divisors that are small or simple, e.g., 2, 3, 4, 5, but with larger numbers, e.g., 6, 7, 8, 9, 11 and 12, children often do more rapid and more accurate work with crutches than without them. To deny certain

children the use of auxiliary figures in division by 7, 8, 9, 12, is to force them into a mode of work which is slower and less accurate for them. Teachers must watch children and encourage mechanical operations without auxiliary figures shown in the accompanying solution, 0652. But when

$$9\overline{)\,5868}^{\,41}$$

children are found whose rate of speed and whose accuracy would both be reduced by denying them the use of the crutches 0 in the quotient and the small 4 and 1 in the dividend, they should not be made victims of fixed pedagogical procedures. Auxiliary figures in multiplication are not of any measurable worth; in division and in subtraction they contribute materially to the convenience and the confidence of some types of workers.

Division by a Number of Two Orders.—1. *Gradation.*—Instead of using the sequence 11, 12, 13, 14, 15 . . . 19, 21, 22, 23, 24 . . . 28, etc., as the numerical order of divisors, it has been found more advantageous to use **11, 12, 21, 22, 31, 32, 41, 42, 19, 29, 39,** first and 24, 36, 47, etc., later. This rearrangement is due to the fact that the divisors in heavy type are so close to 10, 20, 30, 40 that they present less difficulty to the children when they seek an approximate quotient.

A second factor in gradation at this point is the selection of the dividend. The suggested sequence is, therefore,

$$21\overline{)231} \qquad 21\overline{)294} \qquad 21\overline{)105} \qquad 21\overline{)1071}$$

instead of using $21\overline{)1846}$ as the first type of example. The reasons for this general gradation will become apparent to the teacher after each example is solved.

2. *Rationalization.*—No attempt should be made to give explanations or justifications for the procedure in this more advanced division. The children should learn it deductively and by continuing the process followed in the simpler di-

vision. The old distinction between short and long division, nonexistent in mathematics, with changes in algorism, should not be continued. To teach children to divide by 21, 22, 31, 32 or 29, we continue the process begun earlier. To illustrate: the child divides 278 by 8 thus: $34\frac{6}{8}$, $8\overline{)278}^{3}$ 4698 by 9, thus, $9\overline{)4698}^{11}$ 522. But when the divisors are raised to 12 it may be necessary, in the interests of accuracy and speed, to permit the following long form instead of the shorter one

Long form	Short form
489	
12)5868	
48	
106	4 8 9
96	$12\overline{)5868}^{10\,10}$
108	
108	

The children are shown that in the long form the carrying, multiplying or subtracting is performed on paper rather than "all kept in mind." The divisor may now be increased to 21 and the child is shown that the process does not vary, e.g.,

11		$279\frac{9}{21}$
21)231	or	21)5868
21		42
21		166
		147
		198
		189
		9

All of division should be presented as a uniform process, carefully graded and drilled on daily. Most teaching difficulties in division arise from the introduction of artificial forms and distinctions or from the presentation of more than one difficulty in each new step.

Approximating Quotients.—A far-reaching factor in determining efficiency in division is the ability to approximate the correct quotient digit in a reasonable time. Drills must be devised, therefore, which will tend to give the child that experience with numbers that develops ability to find a partial quotient with a minimum of trials. One device is to require children to find partial and approximate quotients in examples like the following:

21)130	21)160	21)180	21)190	21)210
19)130	19)160	19)180	19)190	19)210
49)150	49)250	49)200	49)350	
49)140	49)259	49)143	49)160	49)361
54)163	54)284	54)381	54)425	

At first the children feel helpless; but as the approximate numbers are suggested, 20 instead of 19 or 21, 50 instead of 49 or 54, etc., they acquire the insight into number and the " knack " which comes from experience in arithmetical operations. Children are likewise taught not to use the first figure of the divisor to obtain a quotient; thus, 29)4156 or 295)82467, 2 into 4 is 2, but this is sure to be wrong, better, 30 into 41; likewise 2 into 8 is 4, better, 300 into 800 gives 2.

A simpler drill in approximating quotients is found in the following tabular arrangement used by some teachers:

Divisor 4

4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31
32	33	34	35
36	37	38	39

Divisor 7

7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34
35	36	37	38	39	40	41
42	43	44	45	46	47	48
49	50	51	52	53	54	55
56	57	58	59	60	61	62
63	64	65	66	67	68	69

Teacher points to 33, child sees 32, says 8 and 1 over.

Teacher points to 27, child sees 24, says 6 and 3 over.

Teacher points to 34. Child sees 28, says 4 and 6 over.

Teacher points to 60. Child sees 56, says 8 and 4 over.

The Austrian Method of Division.—The method of subtraction by addition leads to a method of division in which the additive and the multiplicative processes are emphasized. The Austrian method in division does not enjoy the popularity of the Austrian method of subtraction, but this is due more to a prevailing opinion that it is more difficult to learn than to actual evidence based on careful experimentation. Many favor this process of division because it is shorter and combines processes. But in the absence of scientific study as to the relative worth of this mode of division we must suspend judgment as to its desirability. The Austrian mode of division is exemplified below:

```
        421
326)137246
    1304
     ---
      684
      652
      ---
       326
       326
       ---
```

```
        421
326)137246
      684
       326
```

Old or usual form of division with remainder emphasized throughout the process.

Austrian Method of Division. Process: $4 \times 6 = 24$; what added to 4 gives a 2? an 8; put down 8 and carry the 3, $24 + 8 = 32$.

$4 \times 2 = 8$; $8 + 3 = 11$; what added to 1 gives a 7? a 6; put down 6 and carry the 1, $11 + 6 = 17$.

$4 \times 3 = 12$; $12 + 1 = 13$; what added to 13 gives 13? Ans. 0, hence no other number need be put down.

Now bring down the 4 of the dividend; 326 into 684, 2.

$2 \times 6 = 12$; what number added to 2 gives a 4. Ans. 2; put down the 2; carry 1, $12 + 2 = 14$.

$2 \times 2 = 4$; $4 + 1 = 5$; what added to 5 gives an 8? Ans. 3. Put down 3; nothing to carry $2 \times 2 = 4$, $4 + 1 = 5$.

$2 \times 3 = 6$; what added to 6 gives a 6? Ans. 0.

Now bring down 6; 326 into 326, 1 and process is completed.

The process by the Austrian system requires less writing and combines both multiplication and subtraction. Whether this leads to economy of time and increase in accuracy remains to be demonstrated by actual achievement. Writing fewer figures and combining two processes may not only

require more mental effort and more time on the part of the child, but may also cause a greater number of errors in computation.

Quotition and Partition.—Division may take one of two forms. Let us assume that a given whole must be divided into a number of groups; it may become necessary to find the number of groups or it may become necessary to find the number of units in each group. If 16 cents is to be divided among a number of children, each receiving 2 cents, then it is necessary to find the number of children who will be given 2 cents each, but if 16 cents is to be distributed among 8 children, then it is necessary to find the number of cents that will be given to each child. These two forms of division are known as quotition and partition respectively. In quotition the number of equal groups is sought; the numbers in the dividend and in the divisor refer to the same commodity; the numbers in the dividend and in the quotient refer to different commodities. In partition, the number of units in each equal group is sought; the number in the dividend and in the quotient refer to the same commodity; the number in the divisor is abstract. Partition is a form of division more closely related to finding fractional parts of a number, e.g., "$\frac{1}{3}$ of $15"; quotition is a form of division more closely related to problems which find what fractional part one quantity is of another, e.g., "$3 is what part of $15?" While these two forms of division must be understood by the teacher and should be given to the class in the variety of examples offered for solution, these terms should never be taught to the children.

The More Complicated Forms of Division.—The very difficult divisors have not been considered here. Careful gradation will tend to make them less troublesome and constant practice will simplify them. Both the gradation and the drills for the advanced forms of division can be found in any good text. The introduction of ciphers in the divisor

or the dividend, e.g., 204)76058, does not complicate the process as it does in multiplication. If the type of divisor which ends in one or more ciphers is postponed until decimal processes are understood, the ciphers will tend to simplify the division. Instead of dividing $2478.54 by 800, the decimal operation may be introduced and the divisor may be reduced to 8, e.g., $8.00_x)\$24.78_x54$, thus saving time and effort and writing.

SIGNS AND TERMS IN FUNDAMENTAL OPERATIONS

There is an important number of signs and terms, the significance of which children must learn. Chief among the former are $+$, $-$, $\times$, $\div$, $=$; among the latter we must consider *sum*, *minuend*, *subtrahend*, *difference*, *multiplicand*, *multiplier*, *product*, *dividend*, *divisor*, *quotient*, *unit* and *remainder*. The meaning and the function of each should be learned incidentally in the course of solving various examples. Children should not be required to memorize set definitions of any of these. Ability to illustrate and recognize each of these terms and to perform the process indicated by each of these signs should be regarded as sufficient mastery of the symbols in the fundamental operations.

CHAPTER IX

THE FUNDAMENTAL OPERATIONS—(CONCLUDED)]

CHECKS, PROOFS AND SHORT CUTS

Importance of Checking Results.—Children must be impressed with the absolute standard, viz., one hundred per cent accuracy, demanded by the business world. To encourage the child, the school gives credit for method, for correct planning, for effort, even though these are not consummated in an accurate result in a problem. For this reason children underestimate the importance of the answer in arithmetic. To increase accuracy, it is necessary to know a system of checks and proofs for the various fundamental operations. Checks and forms of proof inculcate a critical attitude toward one's own work. They teach children the art of self-correction; they diminish the number of absurd answers and exalt accuracy in arithmetical computations. For these reasons checks and systems of proof must be made an integral part of arithmetical work.

Methods of Checking and Proving.—We may sum up the methods of testing results in the following outline form:

1. *Approximations.*—When, for some reason, it is impossible or impractical to prove the answer absolutely, we may merely test its reasonableness and judge it as an approximation to the correct answer. Thus, in the example, "Find the interest on $3504.27 for 3 yr. 5 mo. at 3%," it would require far too much time and involve useless computations to prove the answer by "working it backwards." In such situations we assume that the time is 3 yr. and 4 mo. or $3\frac{1}{3}$ yr. and that the money is $3500, not $3504.27. The

approximate interest is therefore 10 per cent of $3500, or $350. It is obvious that $350 or any sum less than $350 must be incorrect. The answer that is a little above $350 is deemed reasonable and the child assumes its correctness. This method of approximation demands sustained thought and tends to eliminate ludicrous answers.

2. *Exact Proofs.*—There is a variety of methods of testing for the absolute accuracy of the result. These are listed below with appropriate illustrations.

(*a*) Reword the problem, making the element to be found a given or a known element.

Solve the new example to ascertain if the figure found is identical with the figure sought.

Illustration: A plot of land mortgaged for 60% of its value is mortgaged for $3600. What is the land worth? Ans. $6000.

Check. Land worth $6000 is mortgaged for $3600. What per cent of the value of the land is covered by the mortgage? Ans. $\frac{3600}{6000}=\frac{3}{5}=60\%$.

(*b*) Ascertain whether the answer meets the conditions of the problem.

Illustration: A retail dealer who usually receives a discount of 15% finds that goods listed for $350 were charged at $280. Is the wholesaler giving the retail merchant a better discount? If so, by how much does the latter gain or lose? Usual discount = 15% of $350 = $52.50. New discount, $70, minus usual discount, $52.50 = $17.50, increased discount.

Test. Usual discount $52.50 + new increase $17.50 = $70 new discount. $\frac{70}{350}=\frac{1}{5}=20\%$ = new rate, hence 5% increase. 5% of $350 = $17.50 = increased discount.

(*c*) Solve by another method.

Goods marked $36.60 per gross are sold to Mr. A at a

discount of 25% and 10%. What is the final cost per gross to Mr. A?

Illustration: First Method: 100%−25%=75%; 10% of 75%=7.5%, hence final cost is 67½% of $36.60=$24.71.
Second Method: ¾ of $36.60=$27.45; 9/10 of $27.45=$24.705.

(*d*) Review work with great care. Very often another method involves not a little extra work and any process of working the problem back to original conditions is even more difficult. In such cases a careful examination of every step in both the solution and the manipulation may be taken as an index of probable correctness if not as an absolute guaranty of the validity of the answer.

(*e*) Check the mechanical processes involved.

Checks for addition:

(1) Add columns in reverse order. If columns were added up, add them down or vice versa. If answers agree, the assumption is that the final result is correct.

(2) Successive subtraction. From the sum take, in successive subtraction, each of the addends. This method is usually too long to be considered seriously, although it is given in many textbooks.

(3) Casting out nines.

542	add 5 and 4 and 2; 11;	cast out 9 of 11, result	2	Add these; 13; cast out 9 of 13, Result is 4.
376	add 3 and 7 and 6; 16;	cast out 9 of 16, result	7	
489	add 4 and 8 and 9; 21;	cast out 9's of 21, result	3	
757	add 7 and 5 and 7; 19;	cast out 9's of 19, result	1	
2164	add 2 and 1 and 6 and 4; 13;	cast out 9 of 13, result	4	

When the final numbers agree, as 4 and 4, the result is assumed to be correct. The criticism of this method is given in the discussion of the following one.

(4) Reducing to single digit.

542	add 5, 4, 2; result 11;	add 1 and 1,	result is 2	Add 2, 7, 3, 1; result 13; add 1 and 3, result 4
376	add 3, 7, 6; result 16;	add 1 and 6,	result is 7	
489	add 4, 8, 9; result 21;	add 2 and 1,	result is 3	
757	add 7, 5, 7; result 19;	add 1 and 9; 10;		
		Now add 1 and 0:	result is 1	
2164	Add digits of the answer; 2+1+6+4=13; 1+3=4			

Both these methods, i.e., (3) and (4), are too long to be worthy of practical application. They are mere reciprocals of each other. Assume that the answer was 2146, not 2164, and apply either mode of test; 2146 would give $2+1+4+6=13$; cast out 9 or reduce to a digit and 4 is obtained. 2146 answers the test but is incorrect. These two modes of checking are, therefore, not reliable.

(5) Add groups of addends and then find the total result of these partial answers.

a	354	Add $a+b+c$	354		
b	273		273		
c	856		856		
d	977		1483	Add	1483
e	522			and	1888
f	389				
	33	Add $d+e+f$	977		3371 Check
	3371		522		
			389		
			1888		

This is obviously a check, but the time element counts against it.

(6) Re-add but do not carry figures to next order.

354		354	
273		273	
856		856	
11			
1483	check this thus,	13	total of units column
		17	total of tens column
		13	total of hundreds column
		1483	check

This is like the method suggested in (1).

(7) Add all but one addend. To the sum add the addend not included. The total sum is the check number.

a 354	a 354	
b 273	b 273	
c 856	c 856	
d 977	d 977	
e 522	e 522	
f 389	$a+b+c+d+e=$ 2982	
3371	$f=$ 389	
	3371	check

In the final analysis, the first and the last suggestions are the only feasible ones and few practical workers have recourse to plans 2, 3, 4 and 5, so much praised in schoolbooks.

Checks for Subtraction:

(1) Add result and subtrahend to obtain minuend.

(2) Subtract the result from the minuend to obtain the subtrahend.

Checks for Multiplication:

(1) Interchange multiplier and multiplicand to obtain the same product. Similar to No. 1 under addition.

(2) Divide product by either multiplicand or multiplier to obtain the other factor. Method usually inappropriate.

(3) Casting out nines or reducing to a digit.

254 add 2, 5, 4; 11; cast out 9; 2 | $4\times2=8$, cast out 9;
58 add 5, 8, 13; cast out 9; 4 | 8

2032
1270

14732 add 1, 4, 7, 3, 2, 17; cast out 9; 8

254 add 2, 5, 4; 11; add 1 and 1; 2
58 add 5, 8 13; add 1 and 3; 4 } 4×2=8

2032
1273

15762 add 1, 5, 7, 6, 2; 21; add 2 and 1; 3, hence answer is incorrect.
add 1, 4, 7, 3, 2; 17; add 1 and 7; 8, hence 14732 is correct.

For criticisms see No. 4 under addition.

Checks for Division:

(1) Multiply divisor and quotient to obtain dividend.
(2) Divide dividend by quotient to obtain divisor.
(3) Casting out nines and reducing to a digit when there is no remainder in the answer.

SHORT CUTS

Short cuts have not as yet attained their place of importance in arithmetic because teachers are not convinced of their worth. Too often they are regarded as numerical tricks devoid of sound mathematical basis. In reality the values of short cuts are many and far-reaching.

Value of Short Cuts.—1. *Economy of Time.*—The obvious value of a short cut is its ability to reduce the time and the labor required in mechanical processes. But the important fact to remember in problem work is that time and effort so economized make it possible to solve more problems and to give greater practice in the thought phases of arithmetic.

2. *Added Interest in Mechanical Processes.*—In the solution of problems interest is usually experienced in planning the solution and in obtaining the result, but never in the mechanical computations. Short cuts tend to make these

very mechanical operations interesting by introducing into them a variety of ratios and thus injecting into them a rational element. In multiplying 5423 by 279, there is no interest in multiplying by 9, by 7, and by 2, and then adding the partial products. Instead of following the long process, teach children to multiply 5423 by 9 and then to multiply the partial product by 3, because 27 is 3 times 9, thus:

$$\begin{array}{r} 5423 \\ 279 \\ \hline 48807 \\ 146421 \\ \hline 1513017 \end{array}$$

obtained by multiplying 48807 by 3. $(27 = 9 \times 3)$

To discover that 3 times the partial product is the same as 27 times the multiplicand gives an application of the ratio idea of number that lends interest to this mechanical operation. This method applies to multipliers like 648, 546, 549, 729, 819, 459, 728, 497, etc.

3. *A Deeper Insight into Number.*—The preceding illustration shows conclusively that comprehension and application of short cuts give the pupil a deeper insight into number relations and ratios. The child who learns the short method of multiplying by $12\frac{1}{2}$ (add 2 ciphers and divide by 8), and by 125 (add 3 ciphers and divide by 8), has learned all that is implied in the proportion

$$\tfrac{1}{8} : 1 = 12\tfrac{1}{2} : 100 = 125 : 1000.$$

Every short cut gives a deeper understanding of relations existing among numbers and shows more clearly the force of the ratio conception of number.

4. *Greater Skill in Manipulation.*—A value of short cuts which comes as a sequel to the deeper appreciation of number

relations is increased skill in every variety of manipulation. Each illustration already quoted shows clearly that greater dexterity follows in the wake of the use of these short cuts.

Cautions in Introducing Short Cuts.—1. *Not Prematurely.*—Great care must be taken to introduce short cuts only after the long process is thoroughly understood. To teach short cuts before children understand the longer processes may reduce them to mere tricks of computation. A short cut must be rational and contribute to a deeper comprehension of the functions of number. When the short process can be presented as a rational process, it may be taught without the long form.

2. *Short Cut Not a Mere Mnemonic.*—It follows, therefore, that a short cut must not be taught as a mechanical device devoid of mathematical justification. Taught as such it is of little value and leads to erroneous application when the child is confronted by a slightly different condition. The method of testing multiplication by reducing to a digit is an illustration of an arithmetical process, founded either on no mathematical principle or on one incapable of demonstration to the pupils. So, too, multiplying 92 by 94 and obtaining the product by writing 48 to the right and 86 to the left, 8648, is another example of a short cut that should not be taught. The procedure followed is the following: 96×89; take 96 from 100 and 89 from 100, thus obtaining 4 and 11; $4 \times 11 = 44$; now take 4 from 89 or 11 from 96 and obtain 85; the answer is 8544. When the numbers are 88×85, the process becomes confused and the child is lost because the short form of multiplying 96 by 89 is based on no mathematical principle.

3. *Short Cuts Taught to Be of General Application.*—The short form of multiplication by 15 ($\times 10$; $\frac{1}{2}$ of this product; add the two numbers), or by 11 (54×11, write 5, write 4, insert their sum between the 5 and the 4, 594), or by 41, 21, 61 (multiply by 40, 20, 60 and add the multiplicand

to the product), are illustrations of processes that are not worth the teaching. To multiply by 15 or by 11 or by 71 directly will in the end save time and make for greater accuracy. To multiply 67 by 11 by writing 6 and 7 and inserting their sum will give 6137, a product manifestly incorrect; to require children to learn that when the sum of the two digits is greater than 9, the 1 must be added to the tens figure makes multiplication too conscious a process. A short cut that has exceptions should itself be excepted from the list of forms worth teaching.

4. *Short Cuts Worth Teaching Should Become Habit.*— A short cut must not be taught as an end in itself. Too often the teacher assigns an example to the class and adds, "Be sure to use the short cut." Those who solve the example by the long process are usually marked partly or wholly wrong. But when the suggestion is not given to the class, almost all the pupils follow the long process from force of habit. Short cuts are useless unless they are thought of by the children themselves.

To make the use of short cuts habitual, it is necessary to assign very few new ones to each grade. This gives the class opportunity for sufficient drill. To require a teacher to teach almost all short cuts or to assign merely "short cuts" for any grade is a practice which makes it impossible to reduce abbreviated solutions to habit. Teachers must so frame their examples and problems that the numbers will encourage children to use the short cuts they are taught. Thus, instead of asking for the number of gallons of water lost in 254 days at the rate of 536 gallons of leakage per day, the former number may be changed to 246 or 486 or 497 so that the short cut may be employed. Most problems can be utilized for the double purpose, viz., giving practice in the thought phase of arithmetic and in the use of short cuts. Unless the short cut is used from force of habit, it hinders the mind by imposing an additional burden.

PARTIAL LIST OF SHORT CUTS

Addition and *Subtraction.*

1. To add 58 and 27; add 58+20+7; say 58, 78, 85.
2. To take 27 from 85; reverse the process; say 85, 65, 58.

Multiplication.

1. To multiply by
   25, add 2 ciphers and divide by 4.
   $12\frac{1}{2}$, add 2 ciphers and divide by 8.
   250, add 3 ciphers and divide by 4.
   125, add 3 ciphers and divide by 8.
   $16\frac{2}{3}$, add 2 ciphers and divide by 6.
2. To multiply by
   75, add 2 ciphers, divide by 4, multiply by 3.
   375, add 3 ciphers, divide by 8, multiply by 3.
   625, add 3 ciphers, divide by 8, multiply by 5.
   750, add 3 ciphers, divide by 4, multiply by 3.
   $83\frac{1}{3}$, add 2 ciphers, divide by 6, multiply by 5.
   $62\frac{1}{2}$, add 2 ciphers, divide by 8, multiply by 5.
   This short cut is not always worth using, as it may involve as much computation as the long process.
3. To multiply by 10, 100, 1000.
4. To multiply by 729, 546, 549, etc., as explained above.
5. To multiply by a number containing two factors.

524×64 (8×8).

$$\begin{array}{r} \times 8 \\ \hline 4192 \\ \times 8 \\ \hline 33536 \end{array}$$

The long process involves two multiplications and one addition; the short process employs two successive multiplications.

$4632 \times 175 = 4632 \times 25 \times 7$

$4)463200$

$115800 \times 7 = 810600.$

6. To multiply a number of two orders by a number of two orders on the basis of $(a+b) \times (c+d)$.

34
23
782
c b a

$3 \times 4 = 12$; put down 2 (a) and carry 1.
$(3 \times 3) + (2 \times 4) + 1 = 9 + 8 + 1 = 18$; put down 8 ($b$) and carry 1.
$(2 \times 3) + 1 = 7$ (c).

91
32
2912
c b a

$2 \times 1 = 2$ (a).
$(2 \times 9) + (3 \times 1) = 21$; put down 1 ($b$); carry 2.
$(3 \times 9) + 2 = 29$ (c).

N.B. This short cut is of doubtful value, although it is urged in many textbooks. It is worth showing children who are studying algebra, as this applies numerically what they learn in literal computation.

7. To multiply 52 by 48, 36 by 44, 57 by 63, on the basis of $(a+b) \times (a-b) = a^2 - b^2$.

$52 \times 48 = (50+2)\ (50-2) = 2500 - 4 = 2496.$
$36 \times 44 = (40-4)\ (40+4) = 1600 - 16 = 1584.$
$57 \times 63 = (60-3)\ (60+3) = 3600 - 9 = 3591.$

N.B. Same note as under short cut 6, above.

8. To square a number on the basis of $(a+b)^2 = a^2 + 2ab + b^2$.

$13 \times 13 = (10+3)^2 = 100 + 60 + 9 = 169.$

N.B. Same note as under short cut 6, above.

Division.

1. To divide by 10, 100, 1000, etc.

2. To divide by
   25, move decimal point 2 places to left and multiply by 4.
   125, move decimal point 3 places to left and multiply by 8.
   250, move decimal point 3 places to left and multiply by 4.
   $16\frac{2}{3}$, move decimal point 2 places to left and multiply by 6.
   $12\frac{1}{2}$, move decimal point 2 places to left and multiply by 8, etc.
3. To divide by a number which is a multiple of two factors, divide by the two factors. See illustrations under Value of Multiples, Chapter X.

Fractions.

1. To add fractions with numerator 1: $\frac{1}{3}+\frac{1}{5}=\frac{3+5}{3\times5}$.
2. To add any two fractions, e.g., $\frac{2}{3}+\frac{3}{4}=\frac{(2\times4)+(3\times3)}{3\times4}$.
3. To multiply a number by $\frac{3}{4}$, $\frac{7}{8}$, $\frac{2}{3}$, deduct $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{3}$ of the number from itself.
4. To multiply by mixed numbers, e.g., $9\frac{3}{4}$, $7\frac{7}{8}$, etc., multiply by 10 and deduct $\frac{1}{4}$ of the multiplicand or multiply by 8 and deduct $\frac{1}{8}$ of the multiplicand.

Shortened Algorism.

1. In multiplying or dividing numbers by $1\frac{1}{4}$, $1\frac{1}{2}$, etc., it is not necessary to rewrite the numbers; the abbreviated algorism is preferable, e.g.,

$254\times1\frac{1}{4}$ rather than

$$\begin{array}{r} 63\frac{1}{2} \\ \hline 317\frac{1}{2} \end{array} \qquad \begin{array}{r} 254 \\ 1\frac{1}{4} \\ \hline 254 \\ 63\frac{1}{2} \\ \hline 317\frac{1}{2} \end{array}$$

2. When the cost is set at .50, .25, .75, .12½, it is always best to use the fractional part of a dollar, e.g., ½, ¼, ¾, ⅛.
3. In taking a per cent of a number, it is better to use the fractional equivalent in all cases like 37½%, 83⅓%, 75%, 62½%, 12½%, 14⅐%, etc.
4. In examples where the successive steps involve a series of multiplications and divisions the cancellation form is preferable to the performance of each process separately. In the problem, "Find the cost of filling a tank measuring 4×5×3 ft. with gasoline at 21 cents a gallon," there are two procedures:

(*a*) 4×5×3=60 cu. ft.

$\times 1728$

231)103680 cu. in.(448.83$\frac{9}{77}$ gal.

924

1128

924

2040

1848

1920

1848

720

693

27

231

$= \frac{9}{77}$

448.83$\frac{9}{77}$

.21

44883

89766

$2\frac{5}{11}$

\$94.2545$\frac{5}{11}$

\$94.25

(*b*) $\dfrac{4\times5\times\cancel{3}\times1728\times.\cancel{21}}{\cancel{231}} = \dfrac{.60\times1728}{11} = \dfrac{1036.80}{11} =$

.03

$\cancel{77}$

11

\$94.25$\frac{5}{11}$

\$94.25

Application of Ratio Idea.

A ship travels 190 mi. in 6 hrs.; what distance will be covered in 18 hrs.?

(*a*) $\frac{1}{6}$ of $190 = 31\frac{2}{3}$ mi. per hr. $31\frac{2}{3} \times 18 =$ distance in 18 hrs.

(*b*) Ratio method: $18 = 3 \times 6$, hence 190 mi. $\times 3 = 570$ mi.

Any of the illustrations given under the caption of ratio method of solving problems will show the economy of using this process in contrast to the method of unitary analysis.

CHAPTER X

COMMON FRACTIONS

Common or Decimal Fractions First?—Mathematically considered, decimal fractions ought to precede common fractions, because the former are more closely related to the system of notation in whole numbers. For psychological reasons we teach common fractions first. The decimal fraction has so large a denominator that it does not lend itself so readily to objective demonstrations. In the life of the child, the fractions one-half and one-quarter arise naturally, one-tenth, rather infrequently. For these pedagogical reasons, a decimal fraction is regarded almost as an abstraction for the very young child and it is preceded by the common fraction in the sequence of lessons.

Informal Work with Fractions.—While most courses of study begin a systematic study of fractions in the fourth year, it is not necessary to exclude work in common fractions from the first three years. In these early classes fractions are introduced naturally and incidentally through constructive exercises and measurement and comparative processes. Without any formal analysis or explanation, children can give half an apple, half a paper, a quarter of a string, or half of 10 pencils to each of two children who volunteer to sharpen them; in the same way children can be led to add $\frac{1}{2}+\frac{1}{2}$, $\frac{1}{3}+\frac{1}{3}$, $\frac{1}{4}+\frac{5}{4}$, $1\frac{1}{2}+\frac{1}{2}$, $\frac{1}{4}+\frac{1}{2}$, $\frac{1}{4}+\frac{1}{8}$, $\frac{1}{3}+\frac{1}{6}$; to subtract $\frac{1}{3}$ from 1, $\frac{1}{4}$ from $\frac{3}{4}$, $\frac{1}{4}$ from $\frac{1}{2}$; to multiply $\frac{1}{4}$ by 3, $\frac{1}{2}$ by 8; to divide $\frac{1}{2}$ by $\frac{1}{4}$, $\frac{1}{4}$ by $\frac{1}{8}$, etc. All this work is done concretely through diagrams, paper folding, grouping splints or beads, using money, or any available objective aid.

The accompanying diagram, in proper size, can be made the means of finding answers to the following examples:

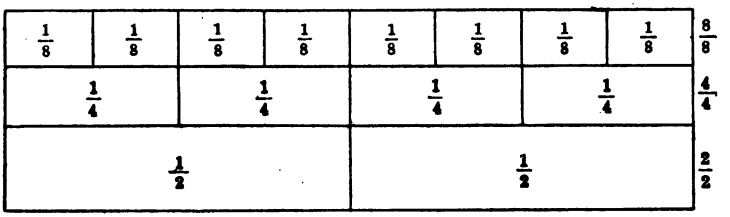


FIG. 11.

$\frac{1}{4}+\frac{1}{2}$	$\frac{7}{8}-\frac{1}{8}$	$\frac{1}{8}\times 3$	$\frac{3}{8}\div 3$
$\frac{1}{8}+\frac{3}{8}$	$\frac{3}{4}-\frac{1}{4}$	$\frac{1}{2}\times 2$	$\frac{3}{4}\div 3$
$\frac{1}{4}+\frac{1}{8}$	$\frac{1}{2}-\frac{1}{4}$	$\frac{1}{4}\times 3$	$\frac{5}{8}\div 5$
$\frac{1}{4}+\frac{3}{8}$	$\frac{1}{4}-\frac{1}{8}$	$\frac{3}{8}\times 2$	$\frac{6}{8}\div 3$
$\frac{3}{4}+\frac{1}{8}$	$\frac{7}{8}-\frac{1}{4}$	$\frac{1}{4}\times 2$	$\frac{1}{2}\div 2$

By using a group of objects children can be led to find $\frac{1}{2}$ of 4, $\frac{1}{2}$ of 8, $\frac{1}{2}$ of 16, $\frac{1}{4}$ of 8, $\frac{1}{4}$ of 4, $\frac{1}{5}$ of 10, etc., until the whole numbers gradually increase to reasonable limits. Problems in finding what fractional part one number is of another can also be given because they can be reduced to concrete terms: Had 4¢, spent $\frac{1}{2}$ of it; what did I spend? Had 4¢, spent 2¢, what part of my money did I spend? Had 8¢, spent $\frac{1}{2}$ of it, what did I spend? Had 8¢, spent 4¢, what part of my money did I spend? When the arithmetical situations are understood, the first of each pair of the above questions may be omitted and the children are asked, " Had 6 marbles and lost 3; what part of my marbles did I lose? " All this work in fractions is informal, free from analysis, rationalization, definitions or rules. It is intensely concrete and often is written in the form, " 1 quarter+2 quarters+3 quarters " until the children learn to focalize on the numerator. In the first three years, an attempt is made to teach the child those computations with fractions that the untaught person learns from actual experience as need arises for " reckoning."

FORMAL WORK IN COMMON FRACTIONS

The Ideas Involved in the Concept Fraction.—When work in fractions is formally introduced, an attempt must be made to build up a rich idea of the meaning of the concept "fraction." The mathematical concept fraction implies at least four distinct ideas: (1) a fraction is one or more of the equal parts of *a unit*; (2) a fraction is *one* of the equal parts of *a number of units;* (3) a fraction is an indicated division; (4) a fraction is only an expressed ratio. Each of these constituent ideas must be taught and stressed.

A Fraction as One or More of the Equal Parts of a Unit.— The child must recognize the fact that $\frac{3}{4}$, $\frac{4}{5}$, $\frac{2}{7}$, imply that *a* unit has been divided into 4, 5, 7 equal parts and that 3, 4, 2 parts, respectively, have been taken. Children should now be required to "tell the story of each fraction," thus:

A man has \$$\frac{1}{2}$; \$$\frac{1}{2}$ tells me that \$1 was divided into 2 equal parts and 1 part was taken.

$\frac{3}{4}$ of a block was paved; $\frac{3}{4}$ tells me that 1 block was divided into 4 equal parts and 3 parts were paved.

Ran $\frac{2}{5}$ of the distance; $\frac{2}{5}$ tells me that a distance was divided into 5 equal parts and 2 parts were covered.

The teacher now sends children to the board and requires them to draw lines, circles, rectangles, and thereby tell "what the fractions $\frac{3}{8}$, $\frac{1}{6}$, $\frac{2}{5}$, tell you." Children are given paper, strings, etc., and told to show "what these fractions tell you." When the drill given is sufficient to make clear to the child the function of the two terms of a fraction, the teacher asks, "What number tells you into how many equal parts to divide the unit? What number tells you how many parts to take?" When the answers, "The number below" and "The number above" are obtained, the teacher then sums up the lesson on the board as follows:

$\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{6}, \frac{3}{8},$ = tells how many equal parts to take / tells into how many equal parts to divide the units = Numerator / Denominator

The children see the function of denominator and numerator long before these terms are taught. The final step in the lesson is to lead the pupils to formulate the meaning of a fraction.

A Fraction One of the Equal Parts of a Number of Units.— The full understanding of a fraction requires a knowledge of the fact that $\frac{3}{4}$ may be regarded as $\frac{3}{4}$ of 1 and $\frac{1}{4}$ of 3; that $\frac{2}{3}$ may be recognized as $\frac{2}{3}$ of 1 and $\frac{1}{3}$ of 2. To make this second relationship clear, children are asked to tell the difference between $\frac{3}{4}$ of 1 ft. and $\frac{1}{4}$ of 3 feet. To illustrate the interpretations, two vertical lines are drawn on the board, one 12 inches, the other 36. Measurements are made, and $\frac{3}{4}$ of 1 foot is found to be 9 inches, while $\frac{1}{4}$ of 3 feet is also 9 inches.

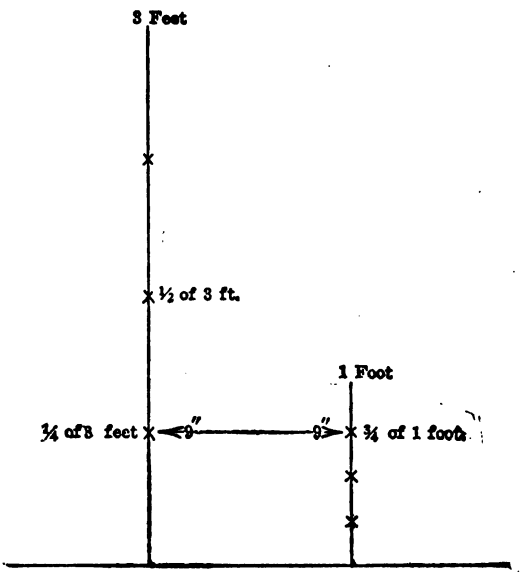


FIG. 12.

In a similar way, by actual paper folding, children are made to discover that $\frac{2}{8}$ of 1 sheet of paper is the same size as $\frac{1}{8}$ of 2 sheets of paper. Fractions are now written on the blackboard and children are required to "tell both stories of each fraction."

$\frac{1}{3}=\frac{1}{3}$ of $1=\frac{1}{3}$ of 4; $\frac{5}{6}=\frac{5}{6}$ of $1=\frac{1}{6}$ of 5; $\frac{7}{10}=\frac{7}{10}$ of $1=\frac{1}{10}$ of 7.

The concept of fraction is thus enriched by a new relationship.

A Fraction as an Indicated Division.—A very important implication in a fraction, viz., that it indicates division, must be indelibly impressed on each child. Much of the later work, not only in common fractions, but in decimal fractions and in cancellation, depends on a clear comprehension of this basic fact. Children can be led to discover this fact by listing on the board fractions and their interpretation as part of a unit or of a group, e.g., $\frac{3}{4}=\frac{1}{4}$ of 3. The teacher then asks, "How can we obtain $\frac{1}{4}$ of 3?" and elicits the directions from the child, "Divide 3 into 4 equal parts." The following arrangement is then placed on the board:

$\frac{1}{2}=\frac{1}{2}$ of 1, i.e., 1 is divided into 2 equal parts, $1\div2$;
$\frac{1}{4}=\frac{1}{4}$ of 1, i.e., 1 is divided into 4 equal parts, $1\div4$;
$\frac{1}{5}=\frac{1}{5}$ of 1, i.e., 1 is divided into 5 equal parts, $1\div5$;
$\frac{3}{4}=\frac{1}{4}$ of 3, i.e., 3 are divided into 4 equal parts, $3\div4$;
$\frac{4}{5}=\frac{1}{5}$ of 4, i.e., 4 are divided into 5 equal parts, $4\div5$;
$\frac{5}{6}=\frac{1}{6}$ of 5, i.e., 5 are divided into 4 equal parts, $5\div6$.

The teacher then leads the pupils to see that 1, 1, 1, 3, 4, 5, are the numerators in each case and that 2, 4, 5, 4, 5, 6, are the denominators. The following tabulation,

$\frac{1}{2}=1\div2=\text{N.}\div\text{D.}$
$\frac{3}{4}=3\div4=\text{N.}\div\text{D.}$
$\frac{4}{5}=4\div5=\text{N.}\div\text{D.}$
$\frac{5}{6}=5\div6=\text{N.}\div\text{D.},$

readily leads to the generalization that any fraction resolves

itself into numerator divided by denominator. Practice must now be given in turning the following into fractions, $5 \div 8$, $8 \div 9$, etc., as well as in converting fractions into division examples.

A Fraction as a Ratio.—The idea that a fraction expresses a ratio should not be taught in any given lesson, but should be developed gradually through experience with fractions, especially in finding what fractional part one quantity is of another. The child who has 2 agates and loses one realizes that 1 of his 2 agates is $\frac{1}{2}$ of his possession. As a result of cumulative experience and maturing insight $\frac{1}{2}$ assumes a new significance, 1 out of 2, 1 as compared with 2, 1 is to 2, 1 : 2. Viewed from the standpoint of mental evolution, it seems reasonable to believe that integers developed from fractional ideas rather than vice versa; fractions are more closely related to measurement of small quantities than integers. The savage who caught one deer while his neighbor brought in two, perceived, although he did not conceive the fact, that he had only one-half of his neighbor's possession. In the measurement process, we perceive the parts before we have estimated the whole; but each part is a fractional equivalent. This ratio in each fraction is conceived only after much experience with such examples as "A has \$4 and needs \$5; what part of the money he needs has he now?" The answer $\frac{4}{5}$ means 4 out of 5; the relation of what he had to what he needs is 4 to 5, hence $\frac{4}{5}$ is equivalent to 4 : 5. The method of teaching the ratio idea in arithmetic is given in greater detail in the chapter on problems.

Reductions of Common Fractions.—The variety of reductions of one form of common fraction to another should be presented concretely at first and then performed by rule deduced from objective experience.

1. *Reduction of whole numbers to fractions.*

   Motive. Give practical situation or appeal to puzzle instinct.

Rationalization. $1=\frac{2}{2}$; $1=\frac{4}{4}$, etc.

$5=\frac{?}{2}=\ominus\ \ominus\ \ominus\ \ominus\ \ominus=\frac{10}{2}$;

$\frac{2}{2}+\frac{2}{2}+\frac{2}{2}+\frac{2}{2}+\frac{2}{2}$

2 halves in 1; in 5, there are 2 halves$\times 5$.

$3=\frac{?}{4}=\oplus\ \oplus\ \oplus=\frac{12}{4}$; 4 quarters in

$\frac{4}{4}+\frac{4}{4}+\frac{4}{4}$

1; in 3, there are 4 quarters$\times 3$.

Rule. To be formulated by the children.

Terms. Proper and Improper Fraction.

2. *Reduction of fraction to whole or mixed numbers.*

Motive. Same.

Rationalization. $\frac{10}{2}=?$ $\overset{1}{\ominus}\ \overset{1}{\ominus}\ \overset{1}{\ominus}\ \overset{1}{\ominus}\ \overset{1}{\ominus}=\frac{10}{2}$ or 5;

$\frac{2}{2}\ \frac{2}{2}\ \frac{2}{2}\ \frac{2}{2}\ \frac{2}{2}$

2 halves in 1; hence $10\div 2=5$.

$\frac{13}{2}=?$ $\overset{1}{\ominus}\ \overset{1}{\ominus}\ \overset{1}{\ominus}\ \overset{1}{\ominus}\ \overset{1}{\ominus}\ \overset{1}{\ominus}\ \overset{\frac{1}{2}}{\cap}$

$\frac{2}{2}\ \frac{2}{2}\ \frac{2}{2}\ \frac{2}{2}\ \frac{2}{2}\ \frac{2}{2}\ \frac{1}{2}$

$=\frac{13}{2}=6\frac{1}{2}$;

$13\div 2=6$ and $\frac{1}{2}$.

Rule. Formulated by class.

Terms. Mixed Number.

3. *Reduction of mixed number to improper fraction.*

Motive. Same.

Rationalization. $3\frac{1}{2}=\frac{?}{2}$

$3\frac{1}{2}=3+\frac{1}{2}$

$=\frac{6}{2}+\frac{1}{2}$

$=\frac{7}{2}$ hence, shorter form $=\frac{3\times 2+1}{\text{halves}}$

Rule. Induced after a number of examples are solved.

4. *Reduction to lower or higher terms.*

This lesson is given in detail in Chapter V as an illustration of an inductive recitation.

Reducing to Lowest Terms by the Greatest Common Divisor.—In actual arithmetical operations there is almost no occasion for finding the greatest common divisor. Where fractions are to be reduced to lowest terms, the method of successive divisions is found to be ample when applied to any fraction likely to arise in business operations. Thus $\frac{14}{42} = \frac{7}{21} = \frac{1}{3}$ is an illustration of the point in question. Few fractions in business computations are as clumsy as $\frac{14}{42}$, and even with such a fraction the greatest common divisor is not necessary. Where the long Euclidean method must be taught, a purely deductive, imitative method should be used and no attempt should be made at rationalization. Even in these large fractions the method of successive factoring saves time.

To teach children to reduce fractions to lowest terms by resolving both numerator and denominator into their factors and then discovering the highest common factor also involves needless waste; this can be seen in the case cited below.

G. C. D. of $\frac{58}{126}$.

$$
\begin{array}{r}
\quad 2 \\
\hline
58)126 \\
\quad 116 \quad 5 \\
\hline
\quad 10)58 \\
\quad\quad 50 \quad 1 \\
\hline
\quad\quad 8)10 \\
\quad\quad\quad 8 \quad 4 \\
\hline
\quad\quad\quad 2)8
\end{array}
$$

G. C. D. = 2.

$\frac{14}{42} = \frac{2 \times 7}{2 \times 7 \times 3}$ The H. C. F. is 2×7, hence

$$
\begin{array}{r|l}
\div 14 & \frac{14}{42} = \frac{1}{3} \\
\div 14 &
\end{array}
\quad \text{or} \quad
\frac{14 \div 14}{42 \div 14} = \frac{1}{3}
$$

Here, again, more time is consumed and the method is unnecessary for the type of fractions usual in commercial and industrial life. A few suggestions for testing divisibility of numbers may be helpful, but these should be pointed out incidentally rather than taught formally and committed to memory:

(1) A number is divisible by 2 if the last digit is even.
(2) A number is divisible by 3 if the sum of the digits is divisible by 3.
(3) A number is divisible by 4 if the last two digits are divisible by 4.
(4) A number is divisible by 5 if the last digit is 0 or 5.
(5) A number is divisible by 6 if it is divisible by 2 and 3 or if it is even and rule (2) applies.
(6) A number is divisible by 8 if the last 3 digits are divisible by 8.
(7) A number is divisible by 9 if the sum of the digits is divisible by 9.

Multiples and Factors.—*Their Importance.*—A knowledge of multiples is necessary in all reductions of fractions, in addition and subtraction of fractions, in cancellation, in proportion, in ratio solutions, in short cuts, and indirectly in most processes in arithmetic. A knowledge of multiples implies a knowledge of factors, for the two have a reciprocal relation. To know that 56 is a multiple of 8 and 7 or that 8 and 7 are factors of 56 enables one to solve division examples by short forms of two successive divisions, thus:

```
   13515.03+
56)756842.00
   56
   ---
   196
   168
   ---
    288
    280
    ---
     84
     56
     --
     282
     280
     ---
      200
```

```
8)756842
7) 94605.25
   13515.03+  Ans.
```

Form A Form B

Form A is more difficult, requires more computations, and more writing of figures. Without a knowledge of factors and multiples interesting interrelations among numbers would never be perceived.

Teaching Multiples and Factors.—To develop the concepts *multiple* and *factor*, write small numbers in a column and ask the class "to give the larger numbers that are exactly divisible by these smaller numbers," thus:

Teacher gives	Children Supply
2, 3,	6, 12, 18, 24, etc.
4, 5,	20, 40, 60, 80, 100, etc.
8, 6,	24, 48, etc.
3, 5, 6,	30, 60, etc.
2, 5, 4,	20, 40, 60, etc.
2, 3, 5,	30, 60, 90, etc.

The teacher then elicits the relationship existing between numbers in the first column and those in the second. When the children formulate the function of each, the teacher writes the words *factors* and *multiples* under the appropriate columns. Children see the function first and then learn the name. The expression "least common multiple," is finally introduced as the best descriptive name for the first multiple in each line.

Finding L. C. M. or L. C. D. by Inspection.—Much valuable time is often lost because teachers interpret the expression "by inspection" to mean, "without any computation whatever," or "by guessing." In seeking the least common denominator for the fractions $\frac{3}{4}$, $\frac{5}{6}$ and $\frac{4}{9}$, children may be seen gazing aimlessly at 4, 6, and 9, hoping to alight by some chance on the least common multiple. They usually select in this accidental way an inappropriate number. "By inspection" must be interpreted to mean "without written computation"; hence, children must be

taught a simple means of determining the least common denominator. The following are some of the means suggested; the first will be found the most rapid device:

1. Select the highest factor and try successive multiples.

   4, 6, 9; 9 is selected; $9\times1, 9\times2, 9\times3, 9\times4=36$. L. C. D., 36.

2. Take the first two numbers and find their L. C. M., then find the L. C. M. of that and the next figure.

   4, 6, 9; L. C. M. of 4 and 6 is 12; L. C. M. of 12 and 9 is 36.

3. Disregard those smaller denominators that are factors of larger ones; then find the L. C. M. of the remaining denominators.

   2, 3, 4, 8, 6; disregard 2 and 4 because of 8; disregard 3 because of 6; find the L. C. M. of 8 and 6; $8\times1, 8\times2, 8\times3=24$. L. C. D. is 24.

4. Write out the successive multiples of each denominator, then select the first multiple that is common to all.

   4, 8, 12, 16, 20, 24, 28, 32, 36, 40
   6, 12, 18, 24, 30, 36, 42, 48, 54, 60
   9, 18, 27, 36 hence 36.

This method often involves the writing of more multiples than are necessary and may, therefore, be longer, but it shows clearly the meaning of *least* and *common*.

To train children to reduce fractions to equivalent fractions that have the same least common denominators, the chart suggested below, hung in front of the class, is found very serviceable. If the fractions are $\frac{2}{3}$ and $\frac{3}{4}$ the child looks for the first column which contains the same denominator for thirds and fourths and finds twelfths; if the fractions are $\frac{5}{6}$ and $\frac{3}{4}$, the child finds that the first suitable column is again the one marked twelfths.

Fractions	The Common Denominators											
$\frac{1}{2}$	$\frac{3}{6}$	$\frac{4}{8}$		$\frac{5}{10}$	$\frac{6}{12}$	$\frac{7}{14}$		$\frac{8}{16}$	$\frac{9}{18}$	$\frac{10}{20}$		$\frac{12}{24}$
$\frac{1}{3}$	$\frac{2}{6}$		$\frac{3}{9}$		$\frac{4}{12}$		$\frac{5}{15}$		$\frac{6}{18}$		$\frac{7}{21}$	$\frac{8}{24}$
$\frac{1}{4}$		$\frac{2}{8}$			$\frac{3}{12}$			$\frac{4}{16}$		$\frac{5}{20}$		$\frac{6}{24}$
$\frac{1}{5}$				$\frac{2}{10}$			$\frac{3}{15}$			$\frac{4}{20}$		
$\frac{1}{6}$	$\frac{1}{6}$				$\frac{2}{12}$				$\frac{3}{18}$			$\frac{4}{24}$
$\frac{1}{7}$						$\frac{2}{14}$					$\frac{3}{21}$	
$\frac{1}{8}$		$\frac{1}{8}$						$\frac{2}{16}$				$\frac{3}{24}$
$\frac{1}{9}$			$\frac{1}{9}$						$\frac{2}{18}$			
$\frac{1}{10}$				$\frac{1}{10}$						$\frac{2}{20}$		
$\frac{1}{12}$					$\frac{1}{12}$							$\frac{2}{24}$

FIG. 13.

L. C. D. Not Found by Inspection.—When many fractions having large denominators must be added, the least common denominator cannot be found by inspection. The long process of finding the least common multiple must then be used. It was suggested in outlining the scope of elementary school arithmetic that such fractions should be excluded, but many courses of study demand these and children must, therefore, be taught how to add $\frac{5}{12}$, $\frac{9}{14}$, and $\frac{7}{18}$. This long process of finding the L. C. D. should be presented in a deductive lesson with rigid exclusion of all rationalization. The teacher must show the process a number of times so that the class will learn through imitation. Any attempt to explain "whys" and "wherefores" will prove only a source of confusion and will interfere with the early mastery of the process.

Addition and Subtraction of Fractions.—*Need of Gradation.*—The most important teaching problem in adding and

subtracting fractions is the matter of gradation. Every step in these two fundamental processes has already been taught, but in applying the known steps care must be taken to introduce one new form at a time. The following gradation may be helpful:

1. Adding fraction with similar denominators:

$$\tfrac{1}{3}+\tfrac{1}{3},\ \tfrac{2}{5}+\tfrac{1}{5},\ \tfrac{1}{6}+\tfrac{4}{6}.$$

2. Adding whole numbers and fractions or mixed numbers:

$$7+\tfrac{2}{3},\ 7+2\tfrac{2}{3}.$$

3. Adding mixed numbers, or fractions, having similar denominators, but the fractions when added producing a total greater than 1:

$$\tfrac{3}{5}+\tfrac{3}{5},\ 1\tfrac{5}{6}+2\tfrac{3}{6}.$$

4. Adding fractions with dissimilar denominators, the total less than 1, with the aid of a diagram such as was suggested for reduction of fractions:

$$\tfrac{2}{3}+\tfrac{1}{6},\ \tfrac{1}{2}+\tfrac{1}{4}.$$

5. Same as Step 4, no diagram, no written or objective aid. L. C. D. by inspection.
6. Adding mixed numbers, the fractions having dissimilar denominators:

$$2\tfrac{3}{4}+1\tfrac{1}{8}.$$

7. Same as Step 6, the fractions when added giving more than 1:

$$1\tfrac{1}{3}+2\tfrac{2}{5}.$$

8. The number of fractions and mixed numbers added increased beyond two.

For subtraction the gradation may be represented in the following exercises:

1. $\tfrac{5}{6}-\tfrac{3}{6}$, similar denominators.
2. $2\tfrac{5}{6}-1\tfrac{3}{6}$, similar denominators and mixed numbers.
3. $\tfrac{5}{6}-\tfrac{1}{4}$, dissimilar but related denominators; use a diagram.
4. $2-\tfrac{1}{8}$, fraction or mixed number from whole number.
   $2-1\tfrac{1}{8}$.

5. $2\frac{7}{8}-1\frac{3}{4}$, denominators dissimilar but related, mixed numbers. No borrowing. L. C. D. by inspection.
6. $2\frac{1}{8}-\frac{3}{8}$, borrowing but denominators similar.
   $5\frac{1}{3}-1\frac{2}{3}$.
7. $4\frac{1}{8}-1\frac{3}{4}$, borrowing, dissimilar denominators.

The Algorism in Addition of Fractions.—There are two accepted forms of work in addition of fractions, and each requires as many cautions and as much care on the part of the child as the other.

$2\mid\frac{5}{6}=\frac{20}{24}$	$2\frac{5}{6}=2\frac{20}{24}$	$2\frac{5}{6}$ | 20
$1\mid\frac{3}{8}=\frac{9}{24}$	$1\frac{3}{8}=1\frac{9}{24}$	$1\frac{3}{8}$ | 9
$3\mid\frac{1}{4}=\frac{6}{24}$	$3\frac{1}{4}=3\frac{6}{24}$	$3\frac{1}{4}$ | 6
$6\quad\frac{35}{24}=7\frac{11}{24}$	$6\frac{35}{24}=7\frac{11}{24}$	$6\quad\frac{35}{24}=7\frac{11}{24}$
Form A		Form B

This form requires the rewriting of the whole numbers or each of the common denominators and often causes errors because in adding the fractions the children carelessly include the denominators; thus 20+9+24+6; 59, $\frac{59}{24}$, etc.

The amount of writing is here reduced, but children often err either by carelessly omitting the denominator after adding the numerators 20, 9 and 6, or by including the 24 in adding 20, 9, and 6. Neither algorism is infallible. The important fact is to have a uniform procedure throughout the school or in a school system.

Algorism in Subtraction of Fractions.—The same two forms that are suggested for addition of fractions may be used in subtraction. In cases where the borrowing process enters, the following form is shorter and requires less auxiliary computation:

1 | 8		2 $\frac{1}{8}$	$-1\frac{3}{4}$
$2\frac{1}{8}$ | 1+8		↓	|
$-1\frac{3}{4}$ | 6	(*a*)	2 ↓	$-1\frac{3}{4}=\frac{1}{4}$
$\frac{3}{8}$	(*b*)	$\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$	

The auxiliary figures are indicated merely to show the number of operations.

$$\begin{array}{r|l} & 18 \\ 6 & \\ 7\frac{5}{8} & 15+18 \\ -2\frac{3}{8} & 16 \\ \hline 4 & 1\frac{7}{8} \end{array}$$

$$\begin{array}{lll} & 7\ \frac{5}{8} - 2\frac{3}{8} \\ & \downarrow \\ (a) & 7\ \downarrow - 2\frac{3}{8} = 4\frac{3}{8} \\ (b) & \quad \frac{5}{8} + 4\frac{3}{8} = 4\frac{7}{8} \end{array}$$

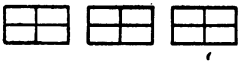
Step (*a*) is very simple and requires little computation;
Step (*b*) applies the skill developed in addition of fractions.
This form allows the use of short cuts and will be found effective when taught to children in the upper grades.

Throughout the teaching of subtraction of fractions, the same form of thought should be followed as was suggested for subtraction of whole numbers. If the Austrian method was used in whole numbers, the additive process must be emphasized in fractions, e.g., $2-\frac{1}{8}$ must read as $\frac{1}{8}+?=2$ as well as $2-\frac{1}{8}=?$

Multiplication of Fractions.—*Gradation.*—In teaching multiplication by a fraction, nice gradations are not found to be helpful. One naturally begins by multiplying (1) whole numbers by fractions; then (2) fractions by fractions; then (3) whole numbers by mixed numbers; and finally (4) mixed numbers by fractions or mixed numbers. To simplify the process of multiplication of fractions, the "of" form should be used first.

1. Fraction and Whole Number.—In teaching $\frac{1}{3}\times 2$, $\frac{3}{4}\times 3$, etc., it is found helpful to present the new form diagrammatically and through addition. Thus,

$\frac{1}{3}\times 2=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$. But $\frac{1}{3}\times 2=\frac{2}{3}$.



$\frac{3}{4}\times 3=\frac{3}{4}+\frac{3}{4}+\frac{3}{4}=\frac{9}{4}$; $\frac{3}{4}\times 3=\frac{9}{4}$. After repeated examples the children are led to generalize that the numerator must be multiplied by the whole number for the new numerator, while the denominator remains unchanged. It now remains to show the children that $\frac{1}{3}\times 2=2\times\frac{1}{3}$.

2. Fraction by Fraction.—The diagram presented is like the one used to teach reduction of fractions.

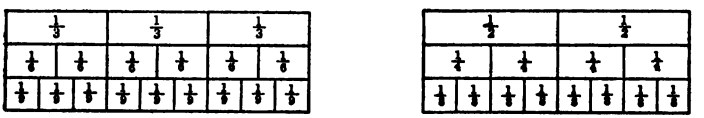


Fig. 14.

$\frac{1}{2}$ of $\frac{1}{2}=\frac{1}{4}$ $\quad$ $\frac{1}{2}$ of $\frac{2}{3}=\frac{1}{3}$

$\frac{1}{2}$ of $\frac{1}{4}=\frac{1}{8}$ $\quad$ $\frac{1}{4}$ of $\frac{1}{3}=\frac{1}{12}$

$\frac{1}{2}$ of $\frac{3}{4}=\frac{3}{8}$ $\quad$ $\frac{1}{3}$ of $\frac{5}{8}=\frac{1}{8}$

In each case the children go to the diagram to find the answer. The teacher then suggests the need of finding the results without the diagram. With this aim in view each example is taken up and subjected to the following analysis:

$\frac{1}{2}$ of $\frac{1}{2}=\frac{1}{4}$; what must be done with the two 1's of the numerators to obtain 1 in the numerator of the answer? If we multiply the two 2's of the denominators what do we obtain? $\frac{1}{2}$ of $\frac{3}{4}=\frac{3}{8}$; what must be done with the 3 and the 1 to obtain 3 for the numerator in the result? What must be done with the 2 and the 4 to obtain 8, the denominator of the result? After such analyses of $\frac{1}{2}$ of $\frac{1}{2}$, $\frac{1}{2}$ of $\frac{1}{4}$, $\frac{1}{2}$ of $\frac{3}{8}$, the class is led to formulate the law and to test it by applying it to $\frac{1}{2}$ of $\frac{2}{3}$, $\frac{1}{4}$ of $\frac{1}{3}$, $\frac{1}{3}$ of $\frac{5}{8}$.

A few days later, the multiplication form is presented with the aid of the diagrams. I have $\$\frac{1}{2}$, my friend has $\frac{1}{2}$ times as much; what has he? If I have $\$\frac{1}{2}$, and my friend has $\frac{1}{4}$ times as much, what has he? etc.

$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$ $\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}$
$\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}$ $\frac{4}{5}\times\frac{1}{4}=\frac{1}{5}$
$\frac{3}{4}\times\frac{1}{2}=\frac{3}{8}$ $\frac{5}{6}\times\frac{1}{5}=\frac{1}{6}$

The class is now shown the similarity in computation between $\frac{1}{2}$ of $\frac{1}{2}$ and $\frac{1}{2}\times\frac{1}{2}$ and is led to formulate the law for multiplication of fractions by fractions. The law is tested by applying it to $\frac{4}{5}\times\frac{1}{4}$ which yields $\frac{4}{20}$; this when reduced becomes $\frac{1}{5}$. The teaching of steps (*c*) and (*d*), i.e., the multiplication of a whole number by a mixed number and the multiplication of mixed numbers by mixed numbers, presents no new problem. In each type of multiplication by fractions, sufficient practice must be given until the process is habituated for every variety of forms in fractions.

Cancellation.—*Gradation.*—An important requirement in teaching the cancellation process is careful gradation of the various forms and successive difficulties. Briefly these may be listed in the following sequence:

(*a*) $\frac{49}{56}=\frac{7\times7}{8\times7}\quad\frac{\div7}{\div7}=\frac{7}{8}$ — Merely application of reduction to L.T.

(*b*) $\frac{5}{8}\times\frac{3}{5}$, $\frac{5}{8}$ of $\frac{3}{5}$, $\frac{5\times3}{8\times5}$ — The numbers to be canceled are both seen (5 and 5) and the results are (1 and 1).

(*c*) $\frac{2}{5}\times\frac{1}{6}$, $\frac{2}{5}$ of $\frac{1}{6}$, $\frac{2\times1}{5\times6}$ — The factor is seen, 2, but the results are not 1 and 1 as above.

(*d*) $\frac{2}{3}$ of $\frac{3}{6}$, $\frac{2}{3}\times\frac{3}{6}$, $\frac{2\times3}{3\times6}$ — Combination of steps (*c*) and (*d*).

(*e*) $\frac{3}{10}$ of $\frac{4\times5}{6\times8}$, $\frac{3}{10}\times\frac{4}{6}$ of $\frac{5}{8}$ — Conditions of (*d*) applied to many fractions.

(*f*) $\frac{4}{7}\times\frac{5}{6}$, $\frac{4}{7}$ of $\frac{5}{6}$ — The factor 2 is not seen in either numerator or denominator.

(*g*) $\frac{4}{5}\times\frac{5}{6}$ of $\frac{9}{10}$ — Application of previous conditions to more than two fractions.

(*h*) $1\frac{2}{3}\times\frac{1}{2}\times15\times\frac{7}{10}$

(*i*) $\dfrac{\overset{3}{37\frac{1}{2}}\times\overset{2}{16\frac{2}{3}}\times\overset{8}{64}}{\underset{1}{8}\times\underset{1}{8\frac{1}{3}}\times\underset{5}{62\frac{1}{2}}}=\dfrac{48}{5}=9\frac{3}{5}$

Introduction of mixed numbers.

To be taught advanced pupils, in upper grades, as an application of ratio idea in arithmetic.

Mode of Rationalization.—1. Review the law of reduction of fractions to lowest terms, then present $\frac{5}{8}\times\frac{3}{5}$. Not knowing how to cancel, the class solves as follows:

$$\tfrac{5}{8}\times\tfrac{3}{5}=\tfrac{15}{40}=\tfrac{3}{8}$$

2. The teacher then shows that the children performed the following operations:

$$\frac{5}{8}\times\frac{3}{5}=\frac{5\times3}{8\times5}=\frac{15\div5}{40\div5}=\frac{3}{8}.$$

3. The class may now be required to solve,

$$2\times4\div2=?\qquad 2\div2\times4=?$$

$$16\times8\div4=?\qquad 16\div4\times8=?$$

Finding that the results are the same, we elicit: (1) that the division process may be performed before the multiplication; and (2) that it is desirable to perform the division first because this gives smaller numbers to work with, e.g., $16\times8\div4=128\div4=32$, but $16\div4\times8=4\times8=32$.

4. The children are now led to realize that in their solution of $\frac{5}{8}\times\frac{3}{5}$ they followed the longer method; hence the division by 5 should precede the multiplication. They are shown the form,

$$\frac{\overset{1}{\cancel{5}}}{8}\times\frac{3}{\underset{1}{\cancel{5}}}=\frac{1\times3}{8\times1}=\frac{3}{8}\text{ or better }\frac{\overset{1}{\cancel{5}}}{8}\times\frac{3}{\underset{1}{\cancel{5}}}=\frac{3}{8}.$$

5. This rationalization should not be forced. Children who have enough mathematical insight to perceive that

cancellation is an attempt to reduce fractions to lowest terms before multiplying should be led to a full comprehension of the process. It may be found necessary to teach cancellation to other children on the basis of authority and through sheer imitation. Since the aim is to develop habits of rapid and accurate factoring, no undue effort should be made to rationalize this canceling process for all children.

General Suggestions in Teaching Cancellation.—In addition to careful gradation of the various types of situations in canceling, it is necessary to impress the factoring idea or the dividing process on the minds of the pupils. Children should be required to indicate the quotient after each act of factoring. Thus, $\frac{3}{5}\times\frac{5}{8}$ should not be permitted because the longer form $\frac{3}{5}\times\frac{\overset{1}{5}}{8}$ is more desirable. When quotients are not written, children develop the idea that cancellation is "crossing out" and in the example $\frac{3}{5}\times\frac{5}{3}$ they proceed to eliminate all numbers, thus $\frac{3}{5}\times\frac{5}{3}$ and give as the answer, zero. But when the work shows quotients as in $\frac{\overset{1}{3}}{5}\times\frac{5}{\underset{1}{3}}$, the child sees at once that the answer is $\frac{1}{1}$ or 1.

Cancellation must be used in every variety of arithmetical situation. In solving problems by unitary analysis, or by ratio methods, successive multiplications and divisions should always be organized into fractional forms so that cancellations may be used. In problems involving areas, denominate numbers, interest, etc., the cancellation method is usually superior, as can be seen in any illustration.

1. What will $5\frac{1}{3}$ oz. cost if $4\frac{1}{2}$ oz. cost \$34.56?

$$\$34.56 \div \tfrac{9}{2} \times \tfrac{41}{8} = \tfrac{3}{8} \text{ of } \$34.56 \times \tfrac{41}{8}.$$

2. Find the interest on \$2550 for 72 days at 6%.

$$\tfrac{72}{360} \text{ of } \tfrac{6}{100} \times 2550 = \text{Interest.}$$

3. How many jars holding 3 pints each can be filled from a bbl. containing 24 gal.?

$24 \times 4 \times 2 =$ pints in bbl. hence $\dfrac{24 \times 4 \times 2}{3} =$ No. of jars than can be filled.

4. Sold $\frac{3}{4}$ of my stock of 24840 bu. of oats and lost $\frac{2}{3}$ of the shipment in transit. How many bushels finally reached destination? Answer $\frac{1}{3}$ of the shipment, hence

$$\tfrac{1}{3} \text{ of } (\tfrac{3}{4} \text{ of } 24840) = \frac{1 \text{ of } \cancel{3} \text{ of } \cancel{24840}}{\cancel{3} \quad \cancel{4}} = 6210 \text{ bu.}$$

$$1 \qquad\qquad 6210$$

Division by Fractions.—Experience shows clearly and convincingly that the shortest and most accurate form of dividing by common fractions is by inverting the divisor. Whatever method we follow in teaching children to divide by common fractions must ultimately lead to the law of inversion. Various procedures have been evolved for teaching this law to children. We must examine most of these.

Through Reduction to Similar Denominators.—Let us assume that $\frac{2}{3}$ is to be divided by $\frac{2}{9}$. The children are led to formulate the successive steps in the equations, $\frac{2}{3} \div \frac{2}{9} = \frac{6}{9} \div \frac{2}{9} = 6 \div 2 = 3$. Another means of attaining the same end is to rewrite $\frac{2}{3} \div \frac{2}{9}$ as a complex fraction, thus: $\dfrac{\frac{2}{3}}{\frac{2}{9}}$. To simplify this fraction, the $\frac{2}{3}$ must be changed to ninths. $\dfrac{\frac{2}{3} \times \frac{3}{3}}{\frac{2}{9}} = \dfrac{\frac{6}{9}}{\frac{2}{9}}$. The common denominators may now be eliminated and the complex fraction becomes $\frac{6}{2}$ or 3.

This mode of teaching division by a common fraction is clear, simple and comprehensible to children, but it does not teach the most desirable law, viz., that of inversion. In simple fractions like $\frac{2}{3}$ and $\frac{3}{5}$, $\frac{3}{4}$ and $\frac{1}{3}$, this method holds, but when fractions are larger and mixed numbers are introduced, the process of reduction to similar denominators becomes too involved and too costly in time.

Through Analysis of the Division Form.—A second method of teaching division by common fractions is to analyze the steps in dividing by any number, whole or fractional, and then to formulate the process common in all division. The steps in this method are summarized below:

(*a*) $\overline{2)12}$, i.e., same as $\frac{1}{2}$ of 12 or $12 \times \frac{1}{2}$.

(*b*) $\overline{1)12}$, i.e., divisor decreases, quotient increases.

(*c*) $\overline{\frac{1}{2})12}$, i.e., 2 halves in 1, in 12 there are 12×2 or 24 halves, hence $\overline{\frac{1}{2})12}$, same as 12×2.

(*d*) $\overline{\frac{3}{2})12}$, to reduce 12 to halves, multiply 12 by 2,

$\overline{\frac{3}{2})(12 \times 2)}$ halves or

3 halves$\overline{)(12 \times 2)}$ halves, equals $3\overline{)(12 \times 2)}$ or $\overline{3)24}$; hence $\overline{\frac{3}{2})12} = 12 \times 2 \div 3$ or $12 \times \frac{2}{3}$ or $12 \div \frac{3}{2} = 12 \times \frac{2}{3}$.

A comparison is now made of steps (*a*), (*b*), (*c*), and (*d*) and the conclusion, "Invert the divisor," is evolved. Although the process is logical, step by step, it is a method that is too verbal and too abstract to guarantee a favorable reaction by immature pupils. These progressive steps clarify the teacher's ideas on the justification of the law of division, but the children usually do nothing more than accept the con-

clusion and carry out its command to invert the divisor and multiply.

By Reducing the Divisor to One.—A simple rationalization, but one open to the same objections, is found in the method which reduces the divisor to one. The children are shown that to divide by 1 leaves the quantity unchanged, hence $1\overline{)8}$, $7 \div 1 = 7$, etc. To divide $\frac{2}{3}$ by $\frac{3}{4}$, the form $\frac{3}{4}\overline{)\frac{2}{3}}$ is used. The children are now asked to tell by what to multiply $\frac{3}{4}$ to produce 1. If they do not give the correct answer the teacher supplies the information, viz., $\frac{4}{3}$. Since a division example is only a fraction, the divisor and the dividend of $\frac{3}{4}\overline{)\frac{2}{3}}$ may be multiplied by the same number, hence, $\frac{3}{4}\overline{)\frac{2}{3}}$ becomes $\frac{3}{4} \times \frac{4}{3}\overline{)\frac{2}{3} \times \frac{4}{3}}$ and then $1\overline{)\frac{2}{3} \times \frac{4}{3}}$ and finally $1\overline{)\frac{8}{9}}$ or $\frac{8}{9}$. It is evident from the above that $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3}$, hence a few more examples will give a basis for generalizing. But the experienced teacher at once evaluates this method as a means of rationalizing for the exceptionally gifted child in mathematics rather than for the average pupil who grasps each arithmetical reason with effort.

By Objective Demonstration and Parallel Correspondence.—A practical class situation was presented as the introductory step, viz., class badges are to be made from a ribbon $5\frac{1}{4}$ yards long; if $\frac{3}{8}$ yd. is allowed for each, will there be enough badges for all? When some children suggested that the ribbon should be cut into small lengths of $\frac{3}{8}$ yd. each, others objected on the ground that if the number were found insufficient, it would be too late to make the badges smaller. It was then decided "to figure it out first," and the situation was reduced to $5\frac{1}{4} \div \frac{3}{8}$ or $\frac{21}{4} \div \frac{3}{8}$. The children then realized that this was a new problem in arithmetic. To the teacher's question, "What should we learn in arithmetic to-day?" the ready anwer was given, "To divide by a fraction."

A double diagram was presented, each rectangle of sim-

ilar dimension, and the corresponding parts were emphasized by colored chalk. From these diagrams the children were asked to find the answers to the following:

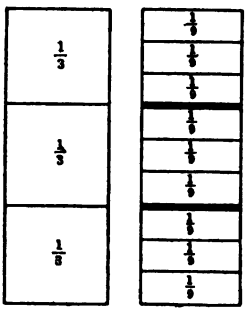


FIG. 15.

$\frac{1}{3} \div \frac{1}{9}$, $\frac{2}{3} \div \frac{1}{9}$, $\frac{2}{3} \div \frac{2}{9}$.

The children found by aid of the diagram

$\frac{1}{3} \div \frac{1}{9} = 3.$

$\frac{2}{3} \div \frac{1}{9} = 6.$

$\frac{2}{3} \div \frac{2}{9} = 3.$

The following facts were elicited: $4 \div 2 = 2$; $\frac{1}{2}$ of 4 or $4 \times \frac{1}{2} = 2$.

$8 \div 4 = 2$; $\frac{1}{4}$ of 8 or $8 \times \frac{1}{4} = 2$.

$8 \div 2 = 4$; $\frac{1}{2}$ of 8 or $8 \times \frac{1}{2} = 4$.

The children were then led to formulate the law that instead of dividing by a whole number we may multiply by the divisor inverted. The class was then asked, "If this holds true for whole numbers, will it hold for fractions?" The children said they did not know, but they would try it on the fractions above, because they knew the correct answers. The next step appeared on the blackboard as follows:

By diagram.	$\frac{1}{3} \div \frac{1}{9} = 3$	$\frac{2}{3} \div \frac{1}{9} = 6$	$\frac{2}{3} \div \frac{2}{9} = 3$
By law.	$\frac{1}{3} \times \frac{9}{1} = 3$	$\frac{2}{3} \times \frac{9}{1} = 6$	$\frac{2}{3} \times \frac{9}{2} = 3$

Every child concluded that the law held true in both fractions and whole numbers and that there was "nothing new in division of fractions, for you turn the divisor upside down and then multiply." The original problem was now brought forth, $5\frac{1}{4}$ yd. $\div \frac{3}{8}$ or $\frac{21}{4} \div \frac{3}{8}$ and it was solved by inverting the $\frac{3}{8}$ and then multiplying, $\frac{\cancel{21}^{7}}{\cancel{4}_{1}} \times \frac{8}{\cancel{3}_{1}} = 21$ badges.

The applications must be varied in form and plentiful in number until the child can divide by common fractions in

any type of problem. The drills should include such examples as $1 \div \frac{1}{2}$, $1 \div \frac{1}{4}$, $1 \div \frac{1}{8}$, $\frac{1}{2} \div \frac{1}{2}$, $\frac{1}{2} \div \frac{1}{4}$, $\frac{1}{4} \div \frac{1}{2}$, $\frac{1}{8})\overline{\frac{3}{4}}$, $5)\overline{\frac{5}{8}}$, $3)\overline{\frac{3}{8}}$, $\frac{1}{8})\overline{\frac{3}{8}}$, $3 \div \frac{1}{8}$, $\frac{1}{8} \div 3$, $1\frac{5}{12} \div 1\frac{2}{3}$, etc. These examples may be put on perception cards and the drills given at regular intervals so that children in the last two years of the school course will not be thrown into hopeless confusion when they are asked to find the answers to $2 \div \frac{1}{4}$, $\frac{1}{4} \div 2$, $\frac{1}{3} \div \frac{1}{6}$, $\frac{1}{6} \div \frac{1}{3}$, etc., without recourse to paper computation.

Gradation in division of fractions is not important. Some texts teach $8 \div \frac{1}{2}$ first, $\frac{1}{2} \div 8$ next, $\frac{1}{2} \div \frac{1}{2}$ third, etc., until the whole series of variations is completed. While nice gradations are necessary in other processes in fractions, they are not especially helpful in division, because the law, once learned, applies to all fractions. It is obvious that introductory examples will exclude mixed numbers, but no other analysis of successive steps of difficulty is necessary.

Complex and Compound Fractions.—Compound fractions, $\frac{2}{3}$ of $\frac{6}{7}$, $\frac{1}{2}$ of $\frac{2}{3}$ of $1\frac{5}{8}$, are useful in practical applications of arithmetic. The discussion on the application of cancellation makes this clear. Complex fractions, which have fractional forms in numerator or in denominator or in both, do not usually arise in common business practices. In teaching the solution of complex fractions a purely deductive and imitative procedure is followed; the pupil is taught to simplify the numerator and the denominator and then translate the complex fraction into division form.

$$\frac{\frac{3}{4}+\frac{5}{6}}{\frac{1}{9}}=\frac{\frac{9}{12}+\frac{10}{12}}{\frac{1}{9}}=\frac{\frac{19}{12}}{\frac{1}{9}}=\frac{19}{12}\div\frac{1}{9}=\frac{19}{\cancel{12}_{4}}\times\frac{\overset{3}{\cancel{9}}}{1}=\frac{57}{4}=14\tfrac{1}{4}$$

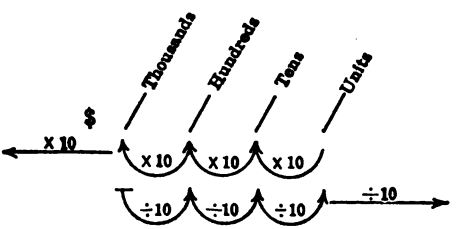
Complex fractions should not be considered part of the minimum course of study and when forced upon teachers by local curricula, they should be taught with a view to economy of time and effort.

CHAPTER XI

DECIMAL FRACTIONS

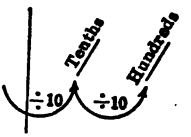
Decimal Notations.—The most important conception in decimal fractions is the system of notation. When the writing and the reading of all decimal forms are understood and reduced to habit, the most significant step has been taken toward the mastery of this branch of arithmetical operations.

It is wrong to assume that decimal fractions have their origin in common fractions or in the notation of our Federal monetary system, although both may be used as aids in understanding the decimal system. From the very first, the decimal fractions must be presented as extensions of the system of notation for whole numbers. The lesson should be informal and begin by reading $1111. The teacher then elicits: (1) the names units, tens, hundreds, thousands, etc.; (2) that each place increases tenfold in going from right to left; and (3) that each place decreases tenfold in going from left to right. The blackboard work at this point presents the following appearance:



The problem is now put: " Where would I write a tenth of a dollar?" Since this is one-tenth of the unit, it obviously

belongs in the first place to the right of the dividing line or in " dimes " place. So, too, a tenth of a dime is put into second place to the right of the line or into the " cents " or hundredths place. The following addition is now made to the preceding boardwork:



For the first few days children write decimals of two places, using a line to separate the whole numbers from the fractions. They are then told to use a " point " instead of a line. When sufficient skill is developed in the notation and numeration of decimals of two places, the thousandths place is taught on the same principle that was suggested above. As the scope of the work increases, the new decimal places are added.

In writing decimals children should be taught: (1) to visualize each number; (2) to recognize instantly the number of places required for a given decimal; (3) to note the number of places occupied by the number as given; (4) to decide on the number of places to be added; (5) to write them continuously from left to right. Thus, if the number *five thousand sixty-four ten-thousandths* is given, the children should: (1) " see " 5064; (2) think " ten-thousandths require four places"; (3) think, " the given number has 4 places"; (4) think, " no extra places are needed"; (5) write a decimal point, a 5, a cipher, a 6 and a 4. To write *five hundred six ten-thousandths*, the child should be trained to: (1) " see " 506; (2) think " four places are required"; (3) think " three places are given"; (4) think " one place is needed"; (5) write, left to right, decimal point, a 0, a 5, a 0, a 6, or .0506. These steps, if followed for a few days, become habit and develop speed and accuracy. To permit

children to write decimals backwards may help them in the initial lesson, but hinders progress by inculcating incorrect habits. Every day must bring its variety of drill lesson in decimal notation. These must include the following:

1. Teacher calls the position of a decimal place, the children give its name.
2. Teacher calls the name of a decimal place, the children give its position.
3. Children read decimals from book, mimeographed sheet, charts, etc.
4. Children write decimals that are spelled out, e.g., "five and seventy-two thousandths."
5. Children write decimals dictated to them.
6. Miscellaneous Drills.
   a. Children change form of decimals without changing their values, e.g., Write $\frac{52}{100}$ decimally as hundredths, as thousandths, as ten thousandths, etc., i.e., $\frac{52}{100} = .52 = .520 = .5200$.
   b. Read .52, .052, .0052.
   c. Write $\frac{62\frac{1}{2}}{100}$ as hundredths, .62½; as thousandths, .625; as ten-thousandths, .6250; as tenths, .6¼, etc. Write ⅓ as tenths .3⅓; as hundredths, .33⅓; as ten-thousandths, .3333⅓, etc.
7. Estimating values of decimals. Which is greater, .01 or .0089, .1 or .0875, .2 or .1897, .5 or .50 or .500? Give in simple form the approximate value of $.429, of $.6110, or .5106 mi., etc.

These drills continue until children have attained a degree of accuracy and speed in decimal fractions comparable with the skill they possess in the manipulation of whole numbers. Time spent on decimal notation is time saved in the later stages of work in decimal forms and in percentage.

Multiplication of Decimals.—There are two modes of presenting multiplication by decimals: (1) through common fractions, and (2) through the analogy that can be pointed out between whole numbers and decimals.

1. *Through Common Fractions.*—Give a problem in which it is necessary to multiply .2 by .4, .2 by .04, .02 by .4, .02 by .04. Elicit (1) that 2 and 4 are multiplied, and (2) that 8 is not the answer. Lead children to perceive that the basic problem lies in finding how to manipulate the decimal points. The blackboard work, at this point, is the following:

.2	.2	.02	.02
$\times .4$	$\times .04$	$\times .4$	$\times .04$
8	8	8	8

Elicit that the correct answer may be found by translating these decimal fractions into common fractions, thus:

$$\frac{2}{10}\times\frac{4}{10}=\frac{8}{100} \quad \frac{2}{10}\times\frac{4}{100}=\frac{8}{1000} \quad \frac{2}{100}\times\frac{4}{10}=\frac{8}{1000}$$

$$\frac{2}{100}\times\frac{4}{100}=\frac{8}{10000}$$

Each of the four " eights " in the first line of work is now changed into .08, .008, .008 and .0008, respectively, all this work, however, being done by the pupils. The aim is now recalled, viz., " to find where to place the decimal point," and by comparing, in each example, the number of decimal places in the product with the number in both multiplicand and the multiplier, the children are led to formulate the law.

2. *Analogy with Whole Numbers.*—The second method of teaching multiplication by decimal fractions has no recourse to common fractions, but carries over the decimal relations of whole numbers to fractional forms. The teacher leads the class to see the parallel of places existing on both sides of the units place, and puts the following on the blackboard:

ten-thousands thousands hundreds tens units • *tenths* *hundredths* *thousandths* *ten-thousandths*

After this corresponding series is carefully noted, the teacher presents multiplication problems in whole numbers of the type of $10\times10=?$ $10\times100=?$ $100\times10=?$ The questions take the following form:

A		*B*
If $10\times10=100$	then	$.1\times.1=?$
If $10\times100=1000$	then	$.1\times.01=?$
If $100\times10=1000$	then	$.01\times.1=?$

Following the parallel that was pointed out before, the pupils readily give the answers .01, .001, and .001, respectively. The teacher then asks: "How many ciphers (in Group *A*), are always contained in the answer?" Since this form of multiplication is so familiar to children in this grade, there is no difficulty in obtaining from them "As many ciphers as there are in both the factors." The inductive step in the lesson is concluded when the teacher's question, "How many decimal places will there be in the product?" brings the answer, "As many places as in both multiplicand and multiplier?" This form of rationalization explains multiplication on the same principle that was followed in teaching decimal notation to pupils.

Gradation and Drills.—No fine gradation is necessary in planning the lesson on the multiplication by decimals. The first examples employ numbers of the type of $.7\times.4$, $.05\times.12$; i.e., decimal fractions without whole numbers. The second type includes mixed numbers like 7.2×1.2, 7.12×1.02, $7.12\times.102$, etc. Any more carefully planned gradation is unnecessary.

Provision must be made for rapid oral drills that seek to reduce "pointing off" to habit. Numerous exercises like the following must be given daily;

.1 × .1	1.2 × .9
.1 × .01	1.2 × 9.
.01× .1	.12 × 9.
.01× .001	.12 × .9
.01×1.	1.2 × .09
.1 ×1.	12. × .9
.01×1.1	12. × .09
.01× .11	.12 × .09
	.012× .009

After the decimal equivalents of the common business fractions are taught, children should be led to perceive that in examples like $712\times\frac{3}{5}$, the decimal multiplication $712\times.6$ is simpler. In the first case, 712 is multiplied by 3 and the product is divided by 5; in the decimal form 712 is multiplied by 6 and one place is "pointed off." So, too, $712\times1\frac{1}{5}$ involves $712\times6\div5$, but 712×1.2 necessitates multiplication by 12 and proper placing of the decimal point. The decimal form by multiplying $712\times\frac{2}{25}$ merely requires the multiplication of 712 by .08, but in the common fractional form 712 is multiplied by 2 and the product divided by 25. Regular and frequent drills by devices already explained will readily develop required skill in the multiplication by decimals.

Division by Decimals.—*The Rationalization.*—Division in which the divisor is a decimal may be performed in a variety of ways. The decimals may be reduced to common fractions and the process carried out by inverting the divisor and then proceeding as in multiplication. If 54.62 is to be divided by .002, the example may be changed to $\frac{5462}{100}\div\frac{2}{1000}$ or $\frac{\cancel{5462}^{2731}}{\cancel{100}}\times\frac{\cancel{1000}^{10}}{\cancel{2}}=27310$. But experience has shown that the simplest form, all possible cases considered, is the method of reducing the decimal divisor to a whole

number and then multiplying the dividend by the same number by which the divisor was multiplied. The mode of teaching division by decimals, presented above, does not lead the child to the most useful rule.

Another method of explaining division by decimals follows a series of steps analogous to those shown in the second method of teaching multiplication by decimals. If $100 \div 10 = 10$, $1000 \div 10 = 100$, $1000 \div 100 = 10$, then by analogy, $.01 \div .1 = .1$, $.001 \div .1 = .01$, $.001 \div .01 = .1$. But from this analysis children find it very difficult to evolve any general rule.

A third method explains division by decimals on the theory that in division the phenomena that occur are the reverse of those in multiplication. The following facts are elicited and illustrated:

In multiplication: Number of places in product = number of places in multiplicand plus number of places in multiplier; i.e., $.02 \times .3 = .006$.

In division: Number of places in quotient = number of places in dividend minus number of places in divisor; i.e., $.006 \div .3 = .02$ or $.006 \div .02 = .3$.

This mode of manipulation of the decimal point, although absolutely correct, may lead to confusion in certain types of numbers, e.g., $276.5 \div .005$; here the child hardly knows how to reduce the seeming difficulty of taking three places from one place. This mode of teaching division by decimals is, therefore, not recommended.

The most effective explanation begins with a practical problem in which the answer is found by dividing \$25.44 by 6. No trouble is experienced by the children, who readily perform the operations of $6\overline{)\$25.44}^{\$\ 4.24}$. The figures in the problem are now changed so that $.6\overline{)\$25.44}^{\$}$ represents the necessary work. A judicious question or two will elicit:

(1) that the decimal point in the divisor is the troublesome factor; and (2) that if .6 were changed to 6., all cause of difficulty would disappear. The children now posit very clearly the aim of the lesson.

The teacher then asks the children to write .6)25.44 in fractional form, and $\frac{25.44}{.6}$ is the result. The next step is to lead children to perceive: (1) that changing the denominator from .6 to 6. multiplies it by 10: (2) that this may be done provided the numerator is also multiplied by 10. The blackboard work now shows:

$$\overline{6)25.44} = \frac{25.44}{.6} = \frac{25.44}{.6} \times \frac{10}{10} = \frac{25_x4.4}{_x6.} = \overline{6)2.544}$$

A few more examples are solved and the children are led to conclude that the divisor is turned into a whole number by multiplying it by 10, 100, or 1000, and that the dividend is then multiplied by the same number. Care must be taken not to have children think or say that the decimal point is moved one place to the right in both divisor and dividend. It must ever be kept before the pupil that he is multiplying both numerator and denominator, dividend and divisor, by the same number.

Drill and Gradation.—The teaching of division by decimals is one of the problems in arithmetic that requires careful analysis of successive difficulties and the formulation of a series of appropriately graded steps. This gradation is given in Chapter IV (page 78) and need not be repeated here.

It remains to point out that as in other fundamental operations, daily drills in rapid oral work are essential for proper habituation. Series of examples of the following type must be used until "pointing off" takes place in the minimum time and with absolute accuracy.

8 ÷2	8÷2	.8 ÷.2	.8 ÷.02
.8 ÷2	8÷.2	.08 ÷.02	.8 ÷.002
.08 ÷2	8÷.02	.008÷.002	.08 ÷.2
.008÷2	8÷.002		.008÷.2
2.08 ÷2			.008÷.02

When division by decimals is properly understood and mechanized, the underlying principle should be applied to division in which the divisor is 1200, 800, 2700, etc. This can be explained on the same basis as that used in decimal divisors, e.g.,

$$246.48 \div 800 = \frac{246.48}{800} \div \frac{100}{100} = \frac{2.46_x48}{8.00_x} = 8\overline{)2.4648},$$

hence divide divisor and dividend by the same number, the number necessary to free the divisor of its confusing final ciphers.

Reduction to Decimal Fractions.—*Importance.*—Reduction of decimal fractions to common fractions involves no teaching difficulty and will therefore be dismissed from consideration. Reductions to decimal fractions are important operations in arithmetic because of the frequency of the need of finding what per cent one quantity is of another. For this reason, reduction to decimal fractions must be emphasized in teaching children decimal operations.

The Rationalization.—A simple mode of teaching the reduction of common fractions to decimal forms is to grade fractions into three groups: Group I, fractions that can be reduced by inspection or through the Federal monetary system; Group II, fractions whose denominators are factors of some decimal denominators; Group III, all other fractions. Each group will be taken up and its mode of teaching will be indicated.

Group I. Fractions like $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{1}{5}$; $\frac{1}{10}$, $\frac{3}{10}$, $\frac{7}{10}$, etc., are included in this class. Children know that a cent is one hundredth of a dollar and are therefore led to conclude that

$\$\frac{1}{4} = \frac{25}{100} = .25$, $\$\frac{1}{5} = \frac{20}{100} = .20$, $\$\frac{3}{4} = 75¢ = \$\frac{75}{100} = .75$, $\frac{3}{10} = 30¢ = \frac{30}{100} = .30$, $\frac{3}{10} = 3$ dimes $= \$\frac{3}{10} = .3\emptyset$, etc.

By this method of inspection a few decimal fractional equivalents are evolved each day and committed to memory.

Group II. In this second class we find fractions like $\frac{7}{125}$, $\frac{31}{50}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{1}{3}$, $\frac{2}{3}$, etc. Children are led to perceive that in reducing $\frac{7}{125}$, $\frac{31}{50}$ or $\frac{5}{8}$ to decimal fractions, the new denominators must be 1000, 100, 100 respectively, hence the following queries are put on the board:

$$\frac{7}{125} = \frac{?}{1000}$$

$$\frac{31}{50} = \frac{?}{100}$$

$$\frac{5}{8} = \frac{?}{100}$$

It therefore becomes necessary to reduce $\frac{7}{125}$ to thousandths and $\frac{31}{50}$ and $\frac{5}{8}$ to hundredths. It is obvious that this can be done by multiplying both terms of the first fraction by 8, of the second fraction by 2, and of the third fraction by $12\frac{1}{2}$. The board work is now increased as follows:

$$\frac{7}{125} = \frac{?}{1000};\ 1000 \div 125 = 8;\ \frac{7 \times 8}{125 \times 8} = \frac{56}{1000} = .056$$

$$\frac{31}{50} = \frac{?}{100};\ 100 \div 50 = 2;\ \frac{31 \times 2}{50 \times 2} = \frac{62}{100} = .62$$

$$\frac{5}{8} = \frac{?}{100};\ 100 \div 8 = 12\tfrac{1}{2};\ \frac{5 \times 12\frac{1}{2}}{8 \times 12\frac{1}{2}} = \frac{62\frac{1}{2}}{100} = .62\tfrac{1}{2}$$

In this way a new series of useful business fractions is reduced to decimal equivalents and the results are committed to memory.

Group III. In the third class we find fractions of the type of $\frac{5}{6}$, $\frac{5}{16}$, $\frac{9}{11}$, or the more unusual one like $\frac{19}{35}$, etc. The methods suggested above obviously do not apply. Since the reduction of such fractions constitutes one of the

necessary mechanical processes in arithmetic, it should be taught with a view towards habituation and therefore with a minimum of rational appeal. The mode of instruction is summarized below:

Rational Form:

Reduce $\frac{5}{16}$ to a decimal.

$$\frac{5}{16} = 5 \div 16 = 16\overline{)5.} \qquad = 16\overline{)5.000}$$

Required form of solution after rational appeal:

$$\begin{array}{r} .31\frac{1}{4} \\ 16\overline{)5.000} \\ 48 \\ \hline 20 \\ 16 \\ \hline 4 \end{array}$$

Any attempt to evolve the process diagrammatically or from an analysis of the conditions involved will usually find children unable to comprehend the succession of reasons. The child who knows that every fraction is equivalent to numerator divided by denominator finds in the equation $\frac{5}{16} = 5 \div 16 = 16\overline{)5.}$ a sufficient explanation to satisfy him. All available time should be spent on habituating this process of reduction.

Decimal Equivalents of Common Business Fractions.— In the course of teaching the reduction of common fractions to decimal forms the child should be drilled on the decimal equivalents of the common business fractions like $\frac{3}{8}$, $\frac{5}{8}$, $\frac{1}{6}$, $\frac{5}{6}$, $\frac{1}{7}$, $\frac{2}{7}$, $\frac{1}{9}$, $\frac{1}{12}$, etc. This work should be incidental. After a few additional common business fractions are reduced to decimal form, they should be subjected to rapid drill. After that, each fraction should be put on a perception card and added to the existing cards which the children know. Each day the cards are taken up in rapid order so that frequent

and regular drills are insured. The fractional equivalents not well mastered should be kept in full view of the class during the day. Five minutes devoted daily to similar rapid drills will soon reduce this information to habit and save much time when percentage problems are introduced. Many of the mechanical devices suggested in the discussion of multiplication tables or the forty-five combinations may serve as effective drills in fixing these decimal equivalents and making recognition of them instantaneous.

CHAPTER XII

PERCENTAGE AND ITS APPLICATION

Introducing the Percentage Concept.—Every teacher realizes that in percentage a new name is applied to a form of arithmetical computation that is well known. But in teaching percentage most teachers present the topic as though it were a new branch of arithmetic only remotely related to processes that the children already know. The best initial lessons on percentage are informal; the new concept is introduced almost incidentally. The first lesson may develop along the following lines:

1. Ask child to solve: Goods cost $500, were sold at a profit of .05, .06, $\frac{8}{100}$, etc.; find the gain. Elicit known mode of solution, viz., multiply and point off two places.

2. Teacher tells class that in business practice the word "hundredth" is not used; neither is it used in class, for no one speaks of having received seventy hundredths in arithmetic. Elicit term "per cent."

3. Ask child to solve the following:

Goods costing $600, were sold at a profit of 5 hundredths or 5 per cent, .07 or 7%, $\frac{8}{100}$ or 8%, etc.; find the gain. Show the sign %.

4. Assign problems in known business applications, using the form, $7\frac{1}{2}$%, 9%, 12%, exclusively.

The children should not be aware that a new division of their work has been introduced, nor should the Latin *per* (by, or out of) and *centum* (hundred) be brought into the lesson. Every added term tends to confuse a concept that would otherwise remain very simple and clear in the children's minds.

Per cent Equivalents of Common Business Fractions.—From the very first, the per cent equivalents of fractions like $\frac{1}{8}$, $\frac{5}{8}$, $\frac{2}{3}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{1}{7}$, $\frac{1}{12}$, $\frac{1}{15}$, $\frac{1}{16}$, etc., must be recognized instantaneously. Children who were taught the decimal forms of common business fractions learn the per cent equivalents in a few short drill lessons. If it is known that $\frac{1}{6}$ is equal to $.16\frac{2}{3}$, an imperceptible effort is required to learn that $\frac{1}{6}$ is also equal to $16\frac{2}{3}\%$. Children should be required to list the three forms in tabular form, thus:

$\frac{1}{4}$	.25	25%
$\frac{3}{4}$	.75	75%

The Cases of Percentage.—The foundational work in percentage is the mastery of the cases, viz., ability to find the per cent of a number, the per cent one number is of another, to find a number knowing the value of a per cent of it. Concretely expressed, these cases are 25% of 120; 30 is what per cent of 120; if 25% of a number is 30, what is the number? Most of the pedagogical problems that arise in teaching these cases of percentage can be traced to the fact that teachers themselves fail to realize that each case in percentage involves no new relationship of numbers and that they, therefore, do not lead children to recognize that the known fractional problems are here expressed in percentage form. In his work in fractions, the child found $\frac{1}{5}$ of 25, what part 5 is of 25, and how many marbles he had originally if, in losing $\frac{1}{5}$ of them, he lost 5. In teaching any of the cases of percentage, problems in the corresponding case in common fractions should be given out first and then the per cent form should be substituted. Thus, in teaching children Case II, the following examples may be used: 9 feet of a 27-ft. pole were broken off. What part of the pole was gone? The class register is 45, class absence, 5; what part of the class is away? A train bound for Albany breaks down after traveling 30 miles. What part

of the 150-mile distance is completed? These problems are now restated and the teacher asks for the per cent of the pole that is broken, the per cent of the class that is absent, and the per cent of the distance that is covered.

In similar manner, the other cases of percentage may be introduced and taught. A summary of these cases, with their corresponding forms, may be helpful:

Fractional Form	*Percentage Form*
Case I.—\$150 must be raised for a poor family; $\frac{3}{5}$ is collected the first day; how much of the necessary sum is now in hand?	Case I.—\$150 must be raised for a poor family; 45% is collected the first day; how much of the necessary sum is now in hand?
Case II.—\$40 was collected for a fund to help a poor family; if \$150 is needed, what part of the money is already assured?	Case II.—\$67.50 was collected for a fund to help a poor family; if \$150 is needed, what per cent of the money is already assured?
Case III.—(*A*) A merchant who sells to gain $\frac{1}{5}$ on what he spent finds that he gained \$12 on a transaction. What did he invest?	Case III.—(*A*) A merchant who sells to gain 12% on what he spent, finds that he gained \$36 on a transaction. What did he invest?
(*B*) A train that completed $\frac{1}{5}$ of its distance has 120 miles still to run. What is the length of the entire trip?	(*B*) A train completed 16% of its trip the first hour. It has 168 miles still to run. How long is the trip?
Goods sold for \$25.50 involved a loss of $\frac{1}{5}$ on my investment. What was the investment?	Goods sold for \$25.80 involved a loss of 14% on my investment. What was the investment?
(*C*) Goods sold for \$25.50 brought a profit of $\frac{1}{5}$. What did they cost?	(*C*) Goods sold for \$14.44 a gross, brought a profit of 12%; find the cost per gross?

This mode of developing the cases of percentage has much to commend it. It is a short and rapid process re-

sulting in economy of time. It involves a logical association and thus integrates the field of arithmetic. It leads to better comprehension, new viewpoints, and more lasting impressions, because of the incidental reviews inherent in this method.

Drills on Percentage Concepts and Relationships.—A comprehension of the cases of percentage, and of the processes of relationships involved, is only the first step in developing the necessary habits in this branch of arithmetic. It is necessary to add to the daily oral work such exercises in percentage as will make the recognition and the solution of these various cases automatic. The forms of drill may be abstract as well as concrete. The following exercises form a partial list of the possibilites that can be devised:

1. Find 1% of \$500, of \$50, \$5000, \$5.
   Find 10% of \$500, \$5000, \$50, \$5.
   Find 1% of 120; 2%, 2½%, 3%, 5%, 6%, 6½% of 120, etc.
   Find 100%, 200%, 150%, 300%, etc., of 25.
2. Find 16⅔% of 12, 37½% of 24, etc.
3. 12 is 16⅔% of what number?
4. 12 is what % of 16?
5. 12 is 20% more than what number?
6. 12 is 14⅔% less than what number?
7. What number is 16⅔% less than 12?
8. What number is 16⅔% more than 12?
9. 16 is what % greater than 12?
10. 12 is what % less than 16?
11. A's age is 40% more than B's. What per cent of A's age is B's.
12. A does a piece of work in 5 days; B does the same work in 8 days. What % does B work slower than A?

The Business Applications of Percentage.—The most practical phase of percentage is its application to the needs of every business situation. Much time and great care must be devoted to profit and loss, trade discount, discounting notes, insurance. commission and brokerage, taxes

and interest, for these sum up the business applications of percentage. There is no need of indicating the method of teaching each one of these applications of percentage, because the teaching problems involved in all of these are very similar in nature. We might, however, suggest general principles that are applicable to all of them.

All Applications of Percentage Repeat the Cases of Percentage.—In teaching each of the business applications, teachers must lead their pupils to recognize the repetition of the cases of percentage. Mathematically considered, interest, commission, and trade discount involve the same operations. In the following problems, the arithmetical relations among the numbers are identical despite any variation in the business situations:

1. \$575 left in a savings bank for 1 yr. and 8 mo. at 4% will earn what interest?
2. An agent sold \$2454 worth of goods in 1 month. What is his commission at the rate of 5%?
3. Goods marked \$24.40 are sold to retailers at a discount of 15%. What does a retailer pay for them?

1. \$575 left in a bank for 2 yrs. earned \$46 in interest. What is the rate of interest?
2. An agent sold \$2554 worth of goods and charged \$127.70 for his services. What is his rate of commission?
3. Goods listed in the catalog for \$28 a dozen are sold to retailers for \$3.20 less. What rate of discount is allowed?

The mathematical identity that exists among all of the business applications of percentage is clearly shown in the accompanying tables. The three elements of percentage, base, rate, and percentage, have their three corresponding elements in each of the applications; the mode of solving each case must therefore be the same. It is a profitable exercise to translate each of the symbolic representations and formulæ in the table into a real problem. It is obvious

Business Applications	Percentage	Profit and Loss	Trade Discount	Insurance	Interest	Commission and Brokerage	Etc.
Elements	Base Rate Percent.	Cost or s. p. Rate of g. or l. Money g. or l.	Listed price Rate of dis. Discount	Policy Rate of ins. Premium	Principal Rate of int. Interest	Goods bought or sold Rate of com. or b'kge. Com. or b'kge.	etc.
Case I	$P = B \times R$	Money g. or l. = c. or s. p. $\times$ % g. or l.	Dis. = listed price $\times$ % discount	Pre = pol. $\times$ % ins.	Int. = prin. $\times$ % int.	Com. or b'kage. = purchase or sales $\times$ % com. or % bkge.	etc.
Case II	$R = \frac{P}{B}$	% g. or l. $= \frac{\text{money g. or l.}}{\text{c.}}$ % g. or l. $= \frac{\text{money g. or l.}}{\text{s. p.}}$	% dis. $= \frac{\text{dis.}}{\text{list price}}$	% Ins. $= \frac{\text{pre.}}{\text{pol.}}$	% Int. $= \frac{\text{int.}}{\text{prin.}}$	% Com. or b'kge. $= \frac{\text{com. or b'kge.}}{\text{purchase or sales}}$	etc.
Case III	$B = \frac{P}{R}$	C $= \frac{\text{money g. or l.}}{\text{\% g. or l.}}$ S $= \frac{\text{money g. or l.}}{\text{\% g. or l.}}$	List price $= \frac{\text{dis.}}{\text{\% dis.}}$	Pol. $= \frac{\text{pre.}}{\text{\% ins.}}$	Prin. $= \frac{\text{int.}}{\text{\% int.}}$	Sales or purchase $= \frac{\text{com. or b'kge.}}{\text{\% com. or \% b'kge.}}$	etc.

FIG. 16.

that this table serves as a summary for the teacher and must not be shown to the children.

Every New Application to be Introduced by a Language Lesson.—The lesson which introduces a new application of percentage must be, essentially, a language lesson. Since all forms of business applications of percentage are mathematically identical, the instructional problem resolves itself into an endeavor to make clear the meaning of the new business terms. In teaching the general subject of interest, the terms, interest, principal, rate, note, discount, etc., must be made clear; so, too, in insurance, the terms, face of policy, premium, life insurance, fire or accident insurance, etc., must be explained with great care and thoroughness. The child who knows the cases of percentage and who has a clear comprehension of the meaning of discount, catalog price, list price, and net price can solve almost any reasonable problem in commercial discount. Too much attention can hardly be given to the language phase in this branch of arithmetic.

Motorize the Business Application of Percentage.—An effective medium of giving children vivid imagery of the actual business procedures in each of the applications of percentage is to dramatize the situations. In teaching stocks and bonds, a company should be organized in class, specimen shares should be written out, shares should be sold at par, at a premium, and at a discount, dividends should be declared and the dissolution actually achieved. Through such motorization a corporate organization becomes real, the *modus operandi* is properly understood, and the business advantage clearly perceived. In teaching insurance, sample policies should be shown and filled out to illustrate the various forms of insurance. So, too, in teaching interest and general elementary banking, children must fill out checks, check stubs, deposit slips, etc.; they must indorse checks to one another, compute interest, make pass book entries—

in a word, every necessary business practice must actually be performed. Private commercial schools boast of classrooms specially equipped to illustrate these various applications of percentage. No elaborate apparatus or furniture is really necessary to make this business procedure appear real and vital to the pupil.

Children Must be Given Reasons for Various Business Practices.—It will not suffice merely to have children understand the terms and the procedure in the business applications of percentage; they must be taught the reasons and the justifications for each business practice that is studied. Children who experienced no difficulty in solving any of the usual examples in interest, when asked why banks pay interest, gave a variety of answers that were interesting because of the ignorance they showed of business practice. Almost all the pupils in the class believed that money deposited in a bank is securely locked in large safes that defy burglars. When these children were asked whether it would not be fair for the depositor to pay the bank for this service, they readily assented. Such ignorance of the reasons why banks pay their depositors makes the topic of interest and all the computations that go with it meaningless number manipulations.

Amusing information awaits the teacher who has not asked pupils why insurance companies pay the full face of the policy in case of death or fire. The following cases were presented to fourteen-year-old pupils: B, who pays a small premium of $15.50 annually for four years, receives $3000 from the fire insurance company when his household goods are destroyed. A, who paid only $120 a year for four years, dies; his family received $5000. The answers that the children gave seemed to show that in their minds an insurance company was an eleemosynary institution composed of individuals of fabulous wealth whose compassion and charity knew no bounds. Evidently no attempt was made by the

teacher to explain the law of averages and chance to the class, to give the rationale of the principles that make insurance possible. Cursory investigation will show that children do not know why agents are paid commission rather than a fixed salary, why goods are listed at one price and sold at a discount, why double discounts, 20 per cent and 10 per cent, are given rather than one large single discount, why banks discount notes, why stocks worth $100 one year are sold at a discount the following year, why people who are older pay a higher rate of insurance, why overhead charges are included in a computation, why certain houses are given lower rates of insurance than others—in a word, business practices, to the majority of our pupils, are devoid of reason and justice.

Indirect Cases Rarely Used in Business Should Not Be Taught in Class.—The plea, so frequently made in current articles on the teaching of arithmetic, that all indirect cases should be omitted in the school, is not based on a knowledge of business needs. These indirect cases arise frequently in actual practice. A business man may know that he cannot sell certain goods for less than $3.20 a dozen and that his retail customers expect the usual trade discount of 20 per cent; his problem is to find that list price which will surely leave $3.20 when the 20 per cent discount is deducted. A publisher who sends out 8254 circulars and receives 743 responses may desire to know what per cent of his appeals brought a response, in order to find whether these circulars are less effective than similar advertising plans. These situations are real and warrant the teaching of indirect cases. We must not, however, ask children to find "what sum of money must be left in the bank for one year eight months, to accumulate interest to the amount of $135.00." Conditions that are either rare or non-existent in the business world should be eliminated from the arithmetic of the schoolroom.

The Mode of Computation of the Business World Must Be Taught in the Class.—It is essential that the mode of computation that has been evolved by business practice, and which those engaged in commercial and industrial life deem most economical, should be taught in school. In teaching profit and loss children should not be taught that gain or loss is always reckoned on the cost; the practice in business often computes the rate of profit on the volume of business, i.e., on the selling price. In the example, "Goods sold for \$12 brought a profit of \$2.00; what is the rate of profit?" the formula $\frac{\text{Gain}}{\text{S.P.}}=\frac{2}{12}=\frac{1}{6}=16\frac{2}{3}\%$ is as correct as $\frac{\text{Gain}}{\text{Cost}}=\frac{2}{10}=\frac{1}{5}=20\%$. Children have repeatedly been marked wrong for computing rate of profit or loss on the selling price instead of on the cost.

The method of computing double discounts is another mathematical situation in which the suggestion to follow business forms of computation might well be adopted. If discount is $33\frac{1}{3}$ per cent and 10 per cent, the simplest solution is to set down "$66\frac{2}{3}$ per cent of catalog price" as the new list price after the first discount; 10 per cent is now allowed on this $66\frac{2}{3}$ per cent, or $6\frac{2}{3}$; deduct $6\frac{2}{3}$ from $66\frac{2}{3}$ and the remainder, "60 per cent of the original catalog price," is the final selling price. This mode of computing is direct and superior in every way to the long process of find the first discount, the new price, the second discount, and then the final price.

In finding the interest on a given sum of money for years, months and days, common business practice follows: (1) the six per cent base; (2) the method of aliquot parts; (3) the method of cancellation; (4) the method of application of a formula; or (5) a convenient base, e.g., 12 per cent, 24 per cent, 36 per cent, 1 per cent. Teachers must find which of these modes is the most prevalent in the local community

and teach that one in preference to the others. These methods are illustrated briefly:

1. *The Six Per Cent Base.*—It is found by experience that economy of time and effort results from the use of a fixed base in computing the interest at any rate. A usual base is 6 per cent. Children learn that the interest on $1.00 for 1 year will be $.06; for 6 months, $.03; for 2 months or $\frac{1}{6}$ year, $.01; for 1 month, $.005; for 10 days or $\frac{1}{6}$ of 2 months, $.00$\frac{1}{6}$; for 6 days, or $\frac{1}{10}$ of 2 months, $.001, etc. If interest is sought at 7 per cent, the answer found by using 6 per cent is increased by $\frac{1}{6}$; if 7$\frac{1}{2}$ per cent, by $\frac{1}{4}$; if 8 per cent, by $\frac{1}{3}$. If interest at 5 per cent is sought, the result obtained by using 6 per cent is decreased by $\frac{1}{6}$ of itself; if 4$\frac{1}{2}$ per cent, by $\frac{1}{4}$; if 4 per cent, by $\frac{1}{3}$; if 3 per cent, by $\frac{1}{2}$, etc. Since 6 per cent is used as a basis in all examples, children become expert in finding interest by short cuts and much time is saved. To illustrate: Since the interest for 60 days at 6 per cent is 1 per cent, we can find the interest on any amount for 60 days at 6 per cent by moving the decimal point two places to the left. The teacher can readily see this economy of labor by solving for himself a few examples in interest.

2. *Method of Aliquot Parts.*—By the method of aliquot parts, the ratios are sought in the time rather than in the rate of interest. Some period, usually a year, is taken as a base, and the additional time is computed on it.

Find the interest on $450 for 2 years, 7 months, 23 days, at 6%.

Interest for 1 year	= $27.00	
Interest for 1 year	= 27.00	
Interest for 6 mos.	= 13.50	($\frac{1}{2}$ yr. hence $\frac{1}{2}$ of $27.00)
Interest for 1 mo.	= 2.25	($\frac{1}{6}$ of int. for 6 mos.)
Interest for 20 days	= 1.50	($\frac{2}{3}$ of 2.25; 20 days = $\frac{2}{3}$ of 1 mo.)
Interest for 3 days	= .225	(3 days = $\frac{1}{10}$ of 30 days, hence take .1 of $2.25)
Interest for 2 yr., 7 mo., 23 days is		$71.475; ans. $74.48.

3. *Method of Cancellation.*—A convenient method of finding the interest on notes that have run a given number of days is to find the interest on the total amount involved for one year, and then to find the interest for the fractional part of a year that actually elapsed. This may be illustrated in the following problem: "A note dated March 18, 1915, for $180 with interest at 6 per cent was paid May 5, 1915. Find the interest and the amount required to settle the account."

Note ran 13 days in March, 30 in April and 5 in May, a total of 48 days. Express 48 days as $\frac{48}{360}$ of a year. The interest on $180 for one year would be $180 \times .06$, hence for 48 days, the interest may be expressed in the following form that encourages cancellation:

$$\text{Int.} = \frac{\$1\cancel{80} \times .0\cancel{6} \times 48}{\cancel{360}\ 2} = \$1.44$$

$180.00 face of note

$181.44 amount due.

4 and 5. *Other Methods of Computing Interest.*—Interest may also be computed by a formula after the process involved and the logic of the various steps have been reduced to habit. The formula is introduced at a point where it sums up in convenient form a procedure that is thoroughly comprehended. Another method of computing interest selects 12 per cent or 36 per cent as the base instead of the prevalent 6 per cent. The steps in the solution are identical with those of the 6 per cent method and the advantages, generally speaking, hold for these new bases as well as for the 6 per cent basis.

It is advisable to teach children to use interest tables in simple and in compound interest. Children must know the value of these tables and the method that was followed in developing them. All good textbooks in arithmetic have reproductions of these tables. The accompanying table is an abbreviated form showing the interest computed by years, days and months. (Fig. 17.) The obvious solution is fol-

Condensed Interest Table. Rate 6%.

Years	$1000	$2000	$3000	$4000	$5000	$6000	$7000	$8000	$9000
1	60	120	180	240	300	360	420	480	540
2	120	240	360	480	600	720	840	960	1080
3	180	360	540	720	900	1080	1260	1440	1620
4	240	480	720	960	1200	1440	1680	1920	2160
5	300	600	900	1200	1500	1800	2100	2400	2700
6	360	720	1080	1440	1800	2160	2520	2880	3240
...	...	...	...	...	...	...	...	...	...

Months	$1000	$2000	$3000	$4000	$5000	$6000	$7000	$8000	$9000
1	5	10	15	20	25	30	35	40	45
2	10	20	30	40	50	60	70	80	90
3	15	30	45	60	75	90	105	120	135
4	20	40	60	80	100	120	140	160	180
5	25	50	75	100	125	150	175	200	225
6	30	60	90	120	150	180	210	240	270
7	35	70	105	140	175	210	245	280	315
8	40	80	120	160	200	240	280	320	360
...	...	...	...	...	...	...	...	...	...

Days	$1000	$2000	$3000	$4000	$5000	$6000	$7000	$8000	$9000
1	.167	.333	.50	.667	.833	1.00	1.167	1.333	1.50
2	.333	.667	1.00	1.333	1.667	2.00	2.333	2.667	3.00
3	.500	1.000	1.50	2.000	2.500	3.00	3.500	4.000	4.50
4	.667	1.333	2.00	2.667	3.333	4.00	4.667	5.333	6.00
5	.833	1.667	2.50	3.333	4.167	5.00	5.833	6.667	7.50
6	1.000	2.000	3.00	4.000	5.000	6.00	7.000	8.000	9.00
7	1.167	2.333	3.50	4.667	5.833	7.00	8.167	9.333	10.50
8	1.333	2.667	4.00	5.333	6.667	8.00	9.333	10.667	12.00
9	1.500	3.000	4.50	6.000	7.500	9.00	10.500	12.000	13.50
10	1.667	3.333	5.00	6.667	8.333	10.00	11.667	13.333	15.00
11	1.833	3.667	5.50	7.333	9.167	11.00	12.833	14.667	16.50
12	2.000	4.000	6.00	8.000	10.000	12.00	14.000	16.000	18.00
13	2.167	4.333	6.50	8.667	10.883	13.00	15.167	17.333	19.50
14	2.333	4.667	7.00	9.333	11.667	14.00	16.333	18.667	21.00
15	2.500	5.000	7.50	10.000	12.500	15.00	17.500	20.000	22.50
16	2.667	5.333	8.00	10.667	13.333	16.00	18.667	21.333	24.00
17	2.833	5.667	8.50	11.333	14.167	17.00	19.833	22.667	25.50
18	3.000	6.000	9.00	12.000	15.000	18.00	21.000	24.000	27.00
19	3.167	6.333	9.50	12.667	15.883	19.00	22.167	25.333	28.50
20	3.333	6.667	10.00	13.333	16.667	20.00	23.333	26.667	30.00
...	...	...	...	...	...	...	...	...	...

From Buker-Felter Arithmetics, Book 3. (Silver, Burdett & Co.)

FIG. 17.

lowed. To find the interest on $875 for 2 years, 8 months and 18 days, at 6 per cent, we find:

The interest on $1000 for	2 years	is $120.00
The interest on $1000 for	8 months	is 40.00
The interest on $1000 for	18 days	is 3.00
The interest on $1000 for	total time	is $163.00

But the amount is $875, not $1000, therefore the interest will be $\frac{875}{1000}$ or $\frac{7}{8}$ of $163.00, which gives,

$$\frac{7}{8} \text{ of } \overset{\$20.375}{\$163.00} = \$142.625 \text{ or } \$142.63.$$

A more elaborate table (Fig. 18) gives the interest at varying rates and for every combination of days and months. To find the interest for 18 days, add the interest for 10 days, 5 days, and 3 days; to find the interest for 8 months, add the interest for 6 months and 2 months; to find the interest for 14 days, add the interest of 10 days, 3 days, and 1 day; to find the interest on $767.50, use the following combination: (Interest on $100×7)+(interest on $10.00×6)+(interest on $7.)+(½ interest on $1.00). A little practice in the use of the interest table soon makes the child expert in finding any time or money combination in a minimum of time.

INTEREST TABLE

Four Per Cent

	$1	$2	$3	$4	$5	$6	$7	$8	$9	$10	$100	$1000
1 day	0	0	0	0	0	0	0	0	0	0	1	11
3 days	0	0	0	0	0	0	0	0	0	¼	3¼	33
5 days	0	0	0	0	0	0	0	¼	¼	¼	5½	.56
10 days	0	0	0	¼	¼	¼	1	1	1	1	11	1.11
1 month	0	¼	1	1¼	1¼	2	2¼	2¼	3	3¼	33	3.33
2 months	¼	1½	2	2½	3½	4	4½	5½	6	6½	67	6.67
3 months	1	2	3	4	5	6	7	8	9	10	1.00	10.00
4 months	1½	2½	4	5½	6½	8	9½	10½	12	13½	1.33	13.33
6 months	2	4	6	8	10	12	14	16	18	20	2.00	20.00
9 months	3	6	9	12	15	18	21	24	27	30	3.00	30.00
1 year	4	8	12	16	20	24	28	32	36	40	4.00	40.00

Five Per Cent

	$1	$2	$3	$4	$5	$6	$7	$8	$9	$10	$100	$1000
1 day	0	0	0	0	0	0	0	0	0	0	1	14
3 days	0	0	0	0	0	0	0	0	0	0	4	42
5 days	0	0	0	0	0	0	0	1	1	1	7	69
10 days	0	0	0	0	1	1	1	1	1	1½	14	1.39
1 month	½	1	1	2	2	3	3	3	4	4	42	4.17
2 months	1	1½	3	3	4	5	6	7	8	8	83	8.33
3 months	1	2½	4	5	6	8	9	10	11	13	1.25	12.50
4 months	1½	3	5	7	8	10	12	13	15	17	1.67	16.67
6 months	2½	5	8	10	13	15	18	20	23	25	2.50	25.00
9 months	3¾	7½	11	15	19	23	26	30	34	38	3.75	37.50
1 year	5	10	15	20	25	30	35	40	45	50	5.00	50.00

Six Per Cent

	$1	$2	$3	$4	$5	$6	$7	$8	$9	$10	$100	$1000
1 day	0	0	0	0	0	0	0	0	0	0	2	17
3 days	0	0	0	0	0	0	0	0	0	1	5	50
5 "	0	0	0	0	0	1	1	1	1	1	8	83
10 days	0	0	1	1	1	1	1	1	2	2	17	1.67
1 month	½	1	2	2	3	3	4	4	5	5	50	5.00
2 months	1	2	3	4	5	6	7	8	9	10	1.00	10.00
3 months	1½	3	5	6	8	9	11	12	14	15	1.50	15.00
4 months	2	4	6	8	10	12	14	16	18	20	2.00	20.00
6 months	3	6	9	12	15	18	21	24	27	30	3.00	30.00
9 months	4½	9	14	18	23	27	32	36	41	45	4.50	45.00
1 year	6	12	18	24	30	36	42	48	54	60	6.00	60.00

From New York City Course of Study

FIG. 18.

CHAPTER XIII

PROBLEMS; ORIGINAL APPLICATIONS

Importance of Problems in Arithmetic.—Problems vitalize arithmetic, give purpose to its computations, and afford a medium for utilizing the skill it develops in children. Problems intensify the disciplinary values of arithmetic and supply motive for accuracy and speed in its mechanical operations. They arouse whatever interest arithmetic can stir in any child and give a quantitative interpretation of social, economic, and civic experiences. For these reasons, the study of problems in arithmetic now challenges our attention.

Meaning of a "Problem."—The term *problem* must be carefully differentiated from *example*, in arithmetic. A problem is a situation coming naturally into the life or experience of an individual and capable of arousing his effort for its solution. An example is an indicated arithmetical procedure. Thus, the following are examples: "$\frac{2}{3}$ of 24 is 50% of what number?" "Reduce 5 bushels, 2 pecks, 3 quarts to pints." "$\frac{5}{8}+\frac{5}{12}+\frac{5}{6}-\frac{3}{8}=?$" "$52\frac{1}{2}\%$ of \$5759.50 $=?$" "What will a load of flour weigh, if each bag averages $98\frac{3}{4}$ pounds?" Each of these arithmetical situations indicates the mode of reasoning and involves mechanical manipulation rather than thoughtful analysis of mathematical relationships. A problem is more than related figures clothed in words. Its purpose is to develop modes of thought rather than to increase skill in handling figures. It is a situation that requires the pupil to make a readjustment of elements and forces; that challenges deliberative activities; that

brings before him a need arising in industrial or commercial life. Illustrations of problems will be offered in the succeeding phases of the discussion.[1]

Characteristics of Effective Arithmetical Problems.—1. *Problems Must Be Real.*—The first essential of an arithmetical problem is its ability to set before the child a real situation. School problems are, too often, artificial and arbitrary assemblages of conditions designed to stir children's mental activities. The class may be asked, "A earned $28.84 this week, which is $16\frac{2}{3}\%$ more than he earned last week; what did he earn last week?" But every teacher knows and most children sense the situation as forced and unnatural. One of the new textbooks on arithmetic offers the following under the caption "Practical Problems": "A dying man decided to leave $\frac{1}{4}$ of his 600 acres to his oldest son; $\frac{2}{3}$ of what he left the oldest, to his second son; $\frac{3}{4}$ of what he left his second son, to his youngest, and the remainder to be divided equally between his wife and his daughter. What did the daughter receive?" Even those unfamiliar with the vagaries of the dying feel certain that the last hour is not given to arithmetical niceties nor to increasing the suspense of the heirs.

2. *The Hypothetical Problem.*—The discussion of the hypothetical problem brings us face to face with the question of the desirability of including conditions that are not real. An imaginary situation may appear more real to the child, and certainly more attractive, than one taken from an activity too familiar in his life. Fiction is often more real than history and fancy often more vibrant with life than fact. A hypothetical problem may stir a child to more intense

[1] A study of problems in arithmetic that is worthy of careful attention by all teachers is "Material for Arithmetical Problems," Bulletin 2 (1914), issued by the Bureau of Reference and Research, Board of Education, New York City, Albert Shiels, Director. This pamphlet contains illustrative problems of every variety and for every school year.

activity and may excite curiosity and the puzzle instinct more rapidly than a business condition. Hypothetical questions may elicit greater expression of self-activity than inane problems of the type of " How many books at $.06 can I buy for $124.48? " or " If one boy walks $2\frac{3}{4}$ miles to school, how many miles do 15 boys walk who live the same distance from the school? " It seems reasonable to permit occasional hypothetical problems that provoke thought and lead to useful habits of manipulation—especially when both the mode of reasoning and the form of the computation can be transferred to real arithmetical situations.

3. *Problems Must Deal with Familiar but Varied Activities.*—It is evident that the conditions set forth in any problem should be about social and business activities with which the child is familiar. This does not necessarily mean that we must limit our problems to marbles, grocery purchases, and the usual monotonous round so often heard in the classroom. A situation, although imaginary, may be so clearly comprehended as to appear real to the child. Problems based on common business procedures, although not seen in the immediate environment of the child, can readily be imaged by children and acquire a semblance of reality. Problems must not be based, however, on unusual procedures in banks or in technical industries that are unknown to the child; their settings must be familiar or capable of being clearly imaged.

4. *Conditions in Problems to Be True.*—In formulating problems it is essential that costs, distances, speeds, capacities, etc., referred to, should approximate the facts of life. Children should not be asked to find the cost of 254 bales of cotton at $2.50 a bale; to find how many miles per hour a train runs that covers a distance of 189 miles in $2\frac{1}{4}$ hours; to find how many square feet of wall and ceiling will be plastered in a room $20 \times 15 \times 9\frac{1}{2}$ feet; to find the cost of 2 tons, 3 hundredweights, 82 pounds at $.52$\frac{1}{2}$ per pound;

to find the premium on a life insurance policy of $3550 at $\frac{3}{8}\%$ insurance; to find how far a man walks in $4\frac{1}{2}$ days who walks 4 miles in one hour. All of these problems are taken from school arithmetics and embody assumptions and conditions that are false; cotton at $2.50 a bale is a ludicrous supposition; a regular train that runs 84 miles an hour or 189 miles in $2\frac{1}{4}$ hours is still a possibility to be achieved in the future; a room must have at least one door and one window; commodities purchased in large bulk are sold by the hundredweight, not by the pound; life insurance policies do not set so fixed a sum as $3550, nor is the premium rate at any set percentage on each dollar.

A typical illustration of the divorce between classroom and business practice is found in the problems that are usually taught under the head of Partnership. The problem reads as follows: A and B enter into partnership, A investing $2000 and B $3000. At the end of the year it is found that the earnings of the business is $4500. What will be the share of each? The usual solution takes a ratio form:

A deposits $2000
B deposits $3000
A and B deposit $5000

A receives $\frac{2000}{5000}$ or $\frac{2}{5}$ of the earnings or $\frac{2}{5}$ of $4500 which is $1800

B receives $\frac{3000}{5000}$ or $\frac{3}{5}$ of the earnings or $\frac{3}{5}$ of $4500 which is $2700

B receives $900 more than A. But it must be assumed that A and B invest equal time, skill, and labor. According to the problem, the only difference in the investments of the two partners is B's additional thousand dollars. For investing $1000 B therefore receives a compensation of $900 or 90 per cent interest. The situation is not only unbusinesslike, but preposterous and even dishonest. What would the business

man's division of the earnings be? He would give to A 6 per cent on his $2000 and B 6 per cent on his $3000, or $120 to the former and $180 to the latter. This money, $120+$180 or $300, would be deducted from $4500 and leave $4200, which would be divided equally between A and B as a form of salary. For investing an added $1000, B receives an added interest of $60, a fair return. Hence B's total share would be $2100+$180 or $2280 and A's, $2100+$120 or $2220.

An effective arithmetical problem gives knowledge which supplements the daily experience by emphasizing its quantitative aspect. The problems cited above are not true to the facts of life.

5. *Language of Problems.*—A problem must be set forth in language that is clear, simple, and attractive. To word a problem correctly requires, among other things, a knowledge of the extent of the pupils' vocabulary, a ready control of a sufficient number of appropriate business or practical settings that are true in every detail, and also a fluency of expression. Formulating problems is, therefore, an exacting composition exercise which many teachers cannot perform during an arithmetic lesson. A good text is, therefore, an agent of great worth in all problem work.

Another language requirement is variety of construction. No set of problems should be so worded that the child learns to divide the first number by the second, the make the second number the denominator, etc. Teachers often give problems that follow the same sentence structure, thus: A man had $15.75 and lost $5.25, what per cent of his money did he lose? A boy who had 84 marbles lost 21; what per cent of his marbles did he lose? An auto tire guaranteed to stand the wear of approximately 12500 miles lasted only for 3000 miles; what per cent of its expected use did the purchaser enjoy? In all of these problems the first number is the denominator and the second the numerator.

The problems must be so varied in sentence structure that the pupil will be forced to think before deciding on the solution of each successive problem. For school authorities to submit model problems to teachers is not always a successful procedure, because stereotyped forms are often the only result.

Types of Problems.—To secure variety in form and language of problems, it is necessary to employ the full complement of types of problems. These may be summed up under the following types:

1. *The Practice Problem.*—New arithmetical situations and new types of solution are developed by isolated or practice problems. These are almost like examples because they give no clearly imaged situation; they are unrelated to other problems and contain a cue word that suggests, if it does not actually tell, the operation that is desired. The following are practice problems:

1. A man who earns $4.25 a day receives how much for a week of 5 days?
2. A man buys pencils at 1 cent each and sells them for 18¢ a dozen. What is his profit on a dozen?
3. At a rate of 70 cents a dozen for buttons, what will 72 buttons cost?
4. A newsboy buys 5 papers for 3 cents and sells 30 papers daily. What is his profit each day?

These problems are introductory and are designed to afford drill on a new mode of computation or solution. It is often difficult to distinguish between a practice problem and an example.

2. *Problems Grouped around One Industry.*—A device frequently employed in formulating problems is to group all given in one lesson or in a series of lessons around one industry. The first group of problems may pertain to the clothing industry, the next is based on athletics, a third deals with problems on the transportation of freight, a

fourth with the lumber industry, etc. There is great danger that such problems will become artificial or dull—artificial because we are forced to invent a graded series of problems around the same setting, dull because the child tires of the repetition of the same situation. Too often problems so evolved and grouped teach more about industries than about arithmetic.

3. *The Narrative Problem.*—The narrative problems differ from the preceding type in that one situation is selected and made the basis of a set of closely related problems. There is no attempt to study any one industry or to confine more than a few problems to the central theme. Thus, the morning newspaper tells of a state tax levy of $20,000,000 on the city. The teacher then formulates a series of problems on this situation, incorporating the following conditions:

(*a*) If the total city tax bill is $200,000,000, what per cent of it is due to the state levy?
(*b*) The city contains about 5,000,000 people; what would be the per capita tax of this state levy?
(*c*) If the tax rate is $2.30 per $100, what is the total value of the taxable real estate?
(*d*) What would the tax rate have been per $100, if not for this state levy?
(*e*) What difference does this state tax make to a man whose property is assessed at $62,000?

Similarly, a new project, a subway, a canal, a new road, or a new school may be selected as the central theme around which five or six interesting and useful problems concerning interest, taxes, dividends, per capita costs, etc., are formulated. These problems give an insight into business and civic affairs and illustrate the possibilities of giving pupils an arithmetical interpretation of experience. The great danger in these closely interrelated and sequential problems is that an error early in the computation may

vitiate all the work which follows. This condition can be obviated by checking the child's answer after each separate problem.

4. *The "Real Situation" Problem.*—Problems need not always be related to the same central situation nor to the one industry. They may be separate and distinct, each referring to its own set of conditions but each positing a real situation of life. "Which is better, to sell 200 lbs. of flour during the week, costing the grocer $10.00, at 6 cents a pound, or to raise the price to 6½ cents and sell only 120 pounds each week?" Here we have a real situation problem for the small grocer. So, too, children in an athletic contest played 8 games and won 5; the nearest competing class played 9 games and lost 3. What is the standing of each class? What will be the standing if each wins the next game? If each loses the next game? The members of the baseball team show the following records:

Player	At Bat	Hits
A	18	8
B	16	4
C	16	6
D	24	8
E	18	4
:	:	:
:	:	:
K	22	3
Total, including those omitted	204	64

What is the per cent scored by each batter? By the whole team? Here we have real situation problems for the average American boy.

5. *Problems without Numbers.*—In teaching the solution of problems, we may, sometimes, be more concerned with clear visualization and accurate comprehension of the

conditions and the development of sequential thought than with the actual manipulation and the answer. In such circumstances, it is advisable to give problems that contain no numbers and merely require children to "tell" the processes to be followed to obtain the answer. The following are examples of this type of problem:

(*a*) What must you know to compute the standing of a team in any competition?

(*b*) How do you find the cost of goods that were sold to gain a certain per cent and that brought a certain sum as profit?

(*c*) Goods can be sold at a low price and a small profit is made; I can sell half the goods for double the profit on each article. How would you find which was more profitable?

(*d*) Goods can be bought at a discount of $52.50 cash if I pay at once, and no discount if I pay in 90 days. How would you find whether it would be better to pay cash or keep my money in the bank for the 90 days?

(*e*) If a room is twice as long as it is wide, and it is 44 feet long, how would you find the area of the floor?

(*f*) A barrel contains 15 gallons, 3 quarts, and 1 pint; what else must you know to find the number of bottles that can be filled from it?

These are only a few suggestions of a host of possibilities for accurate visualization, clear thinking, and close application in reasoning situations in arithmetic that involve no actual computations. As occasional exercises they are of great worth; as regular procedures they degenerate into stereotyped verbal reproductions.

6. *Incomplete Problems* and 7. *Problems Made by the Pupils.*—The final types of problems are those in which children either complete an incomplete mathematical situation or formulate the problem to illustrate a given formula or case. Thus, children may be asked the following:

(*a*) A factory worth $36,600 was insured at a rate of $4.50 per hundred for only $\frac{2}{3}$ of its value. What question may complete this problem?

(*b*) Invent a problem giving the cost of an article after a discount is allowed and require that the list price be found.

(*c*) 54% of my money is $3888. I keep the rest in a savings bank that pays 3½%. How many problems can you make from this fact?

(*d*) What facts must you know to find how long it will take a train to run from New York City to Albany?

(*e*) What facts must you know to find whether it is better to keep money in a savings bank which pays only 3% or to invest in safe stocks? Make a problem on these facts asking this question.

Value of Having Children Complete or Invent Problems.—Completing problems and inventing new ones are forms of drill of great worth. The child usually analyzes a problem that is given him; now he must synthesize elements and relations to bring about a certain situation. Inventing problems gives a new view of number relations that leads to greater comprehension and deeper grasp. It is also a form of application that approximates natural conditions. In business the merchant finds it necessary to compute the rate of profit on his investment. He must gather the necessary figures and formulate his problem, viz., "With given costs of raw material, labor, rent, advertising, and with set total receipts, what is the rate of profit?" These drills in arithmetical inventions are analogous to the synthetic exercises in grammar. In the latter subject we not only expect the child to find an infinitive used as a substantive, but we also require the child to construct sentences which embody a nominative infinitive. Ability to invent problems gives evidence of mastery of the various types.

Sources of Material for Problems.—To list the sources of materials for problems entails an enumeration of almost all the vital activities of life, for problem work in arithmetic correlates with all other subjects and experiences. The setting of problems must come from conditions in: (1) home economy, (2) personal earnings, (3) various occupations and

industries, (4) commerce, (5) games and recreational activities, (6) geography, (7) history, (8) elementary science, (9) drawing to scales, (10) statistics and graphs, (11) school administration, attendance, and class activities, (12) hypothetical situations, (13) social and civic activities, etc. The wider the range of sources from which problems are taken, the surer are they to approximate the needs of everyday life.

Problems from the Textbooks.—It is expected that teachers will collect appropriate problems all the time in order to be able to supplement the material found in textbooks. But it must ever be borne in mind that to formulate acceptable problems entails a considerable draught on the teacher's time and energies. It is hardly fair to expect the teacher to be the author of all or even of most of the problems to be solved by any class. The textbook must discharge this duty in arithmetic as the printed text does in other subjects. No supervisor expects his teachers to invent sentences for analysis, or texts for geography and history. It is the duty of the school to guarantee to the teacher for pupils' use textbooks that are rich in real and vital problems.

STEPS IN SOLVING A PROBLEM

Orthodox advice in the solution of problems counsels that children shall be taught to (*a*) find what is given, (*b*) ascertain what is required, and (*c*) discover the relation between these two conditions. But in the final analysis the third step resolves itself into "to solve a problem, differentiate the known from the unknown and then solve." A more definite procedure must be formulated if children are to develop a mode of attacking and solving arithmetical problems. What steps should the child consciously take when confronted by a problem? The following four steps are submitted as an answer to this question:

I. A clear recognition of what is called for in the problem.
II. Planning the solution.
III. Carrying out the plan.
IV. Checking the answer.

Each of these steps must now be analyzed in order to determine its significance and its procedure.

I. Recognizing the Problem

Children must be taught that the first step in the solution of a problem is to become familiar with its conditions by setting them forth under two captions,

Given:
Find:

This is absolutely necessary if children are to work with a plan and toward a definite purpose. It is a common experience to find children plunging into the heart of a problem without a preliminary analysis of what is given and a clear knowledge of what is to be found. In the midst of the solution, they occasionally become aware of an absence of aim, and confusion follows. Children think with the ends of their pencils. To avoid this irritating condition teachers often tell their children, "Put down your pencils and think." The class sits impatiently a moment until the signal to begin is given. But to command a class to "think" gives them a task so vague as to induce a feeling of helplessness. Children must be told what to think about. To require them to set forth the elements of the problem under the words "Given" and "Find" leads not only to thought and to a familiarity with the conditions in the problem, but also to a recognition of the goal of the problem. Children should not be permitted to proceed with their work without having completed this preliminary step, which should be insisted upon until it becomes habitual.

II and III. Planning the Solution and Executing the Plan

After the problem is clearly grasped, the child must determine the plan that will bring him to the desired answer. These steps should be indicated in some way in the course of the written work. Let us assume that the following problem is to be solved; Steps 1 and 2 would then appear as here set forth.

Mr. L. took a position in New York City at a salary of $2200 a year. After settling in his new home he had $3600 left in a bank that pays 4% interest semi-annually. What is his total income at the end of a year?

Given: Salary $2200, principal $3600, int. 4%, semi-annually.
Find: Total income.
Plan: 1. Find interest for 6 mo.
2. Find interest for 6 mo.
3. Find total income.

It may be objected that writing out such a plan would take no little time, and that in business no such procedure is tolerated. But it must be remembered that so much penmanship should not be required in all grades. After children have acquired the habit of examining a problem to find known and unknown elements, and of planning the work, all written evidence of these mental operations may be discontinued. If the form of work suggested above were followed out in the fifth and the sixth year of the elementary course, children in the seventh year would no longer precipitate confusion by rushing into a problem without a survey of its conditions and a tentative plan for its solution. The pupil in grades below the seventh year is not yet ready to meet the demands of business. It is hoped to inculcate in the early years those habits of thought that will make the child's later work sound.

Methods of Solving Problems in the School Course.— There are a number of modes of solving problems throughout the school course, each having its proper place and time.

These methods will be summed up in tabular form before entering upon the discussion of each.

1. The graphic method.
2. The method of analysis.
   A. By the ratio method.
   B. By unitary analysis.
   C. By algebraic equation.
   D. By proportion.
3. The use of a rule or formula.

1. THE GRAPHIC METHOD.—*When to Use it?*—The graphic method of solving problems and examples should be taught when the type is new or when the conditions of a problem are difficult to imagine. Thus, if the new example is, "A boy had 24 marbles left after losing $\frac{1}{5}$ of what he had. How many had he at first?" the child has never had the indirect case. The conditions must be reduced to graphic form. The original number of marbles from which $\frac{1}{5}$ was lost must be reckoned as $\frac{5}{5}$. These $\frac{5}{5}$ are represented in a diagram, as below:

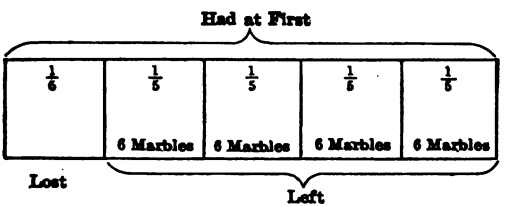


FIG. 19.

It is obvious that $\frac{4}{5}$ of the number of marbles was left. But we know that 24 marbles were left, hence each box or each $\frac{1}{5}$ of the original number is equivalent to 6 marbles. This is found by inspection. Since there are 5 fifths, he must have had 6×5 or 30 marbles.

Similarly: One-fourth of my ruler broke and I had 18

inches left. How long was the ruler? The ruler contains 4 fourths represented in the diagram below:

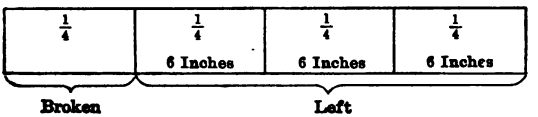


FIG. 20.

$\frac{3}{4}$ of the ruler was left and this measured 18 inches, hence each quarter was 6 inches long (found by inspection) and the entire ruler was 24 inches.

A school allows Grade 6A a plot in its garden measuring 9×18 ft. A path 1 foot wide is run through the plot, parallel to the 18-ft. sides. Each pair of children is given 8 sq. ft. of ground to plant. How many small plots were thus made and how many children planted seeds in this class garden? Here we have a situation that cannot be vis-

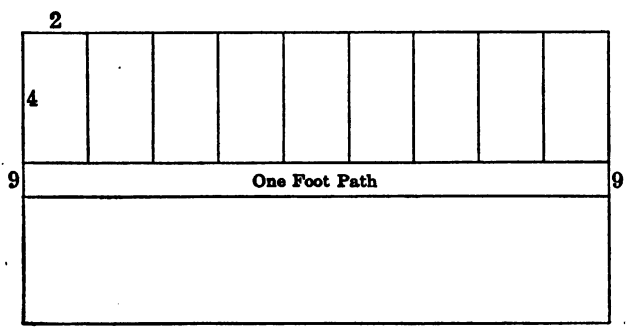


FIG. 21.

ualized by the child unless he draws a rectangle 18×9, and then draws a 1-ft. path leaving two stretches, each 4 by 18 ft. The plot of 8 sq. ft. is obtained by subdividing the 18 ft. length into subdivisions of 2 ft., thus giving rectangular forms 4×2 ft. With the aid of a diagram the con-

ditions of the problem are carried out and the child is led to the answers required.

2. METHOD BY ANALYSIS.—*When to Use It.*—It is obvious that children must learn to solve problems without diagrams. In introducing a new type of problem every graphic means which will translate the relations into concrete terms is welcome. But these objective aids are only means to an end, and must be abandoned after the initial lessons on the new problem. In the problem: "A man can pay only $2550 to his creditors; this is 15 per cent of what he owes them; what does he owe?" we have a type similar to the examples about the marbles and the ruler which were solved graphically. It is obvious that the form of the problem is new but the type is old. In all cases where a type of problem is either understood or where an old type is introduced in a new form, a method of rational analysis and not one of graphic presentation should be used.

Illustration.—In the earlier problem, it was found that $\frac{4}{5}$ of the number of marbles is 24; and the original number was required. The analysis without the diagram becomes,

$\frac{4}{5}$ of the number of marbles = 24

$\frac{1}{5}$ of the number of marbles = $\frac{1}{4}$ of 24 = 6 marbles

$\frac{5}{5}$ of the number of marbles = 6 marbles $\times$ 5 = 30 marbles.

The instructional difficulty is to teach children to understand that if $\frac{4}{5}$ of the number of marbles = 24, then $\frac{1}{5}$ of them is $\frac{1}{4}$ of 24 and not $\frac{1}{5}$ of 24. This can best be accomplished by referring once more to the diagram and using the word "boxes" instead of fifths, thus:

$\frac{4}{\text{Boxes}}$ contain 24 marbles

$\frac{1}{\text{Box}}$ contains $\frac{1}{4}$ of 24 marbles.

But these "boxes" are "parts," hence use the form,

$\frac{4}{\text{Parts}}$ represent 24 marbles

$\frac{1}{\text{Part}}$ represents $\frac{1}{4}$ of 24 marbles.

The term "part" now gives way to "fifths," and we have

$\frac{4}{\text{Fifths}}$ of the number of marbles is 24

$\frac{1}{\text{Fifth}}$ of the number of marbles is $\frac{1}{4}$ of 24.

The final form is now attempted, viz.,

$\frac{4}{5}$ of the number of marbles = 24

$\frac{1}{5}$ of the number of marbles = $\frac{1}{4}$ of 24.

The fractions $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{5}$, $\frac{1}{11}$, $\frac{1}{13}$ are used for drill in order to impress the mind with the fact that the denominator or the name of the part has no bearing on the issue involved. The children find $\frac{1}{7}$ by taking $\frac{1}{4}$; similarly $\frac{1}{8}$, $\frac{1}{9}$, $\frac{1}{11}$ or $\frac{1}{13}$, by taking $\frac{1}{4}$ of the given quantity. In this way the gap between graphic representation and arithmetical analysis is bridged.

In the problem cited above, children are led to see that 15 per cent of the debt is $2550 and that 100 per cent of the debt is sought. The form employed is, therefore,

15% of debt = $2550
1% of debt =
100% of debt =

We must now turn to the various forms of analysis that may be employed.

A. and *B. Ratio Method and Unitary Analysis.*—The prevailing method of analysis of problems in recent years was to compute any quantity in terms of one as the unit. To find the cost of $6\frac{1}{2}$ lb. when $\frac{7}{8}$ of a lb. cost $2.30, or to find how far a train can run in 4 hours and 40 minutes at the rate of 70 miles in 2 hours and 20 minutes, it was deemed necessary to find the cost of 1 lb. then the cost $6\frac{1}{2}$ lb., to find the number of miles covered in 1 hour and then the number of

miles in 4 hours and 40 minutes. With the ratio conception of number came the recognition that 1 need not always be the unit, but that any convenient number might be selected as the basis of computation. The older method is known as "unitary analysis," the more recent form of solution as the "ratio analysis." The two modes of analysis are illustrated in the following parallel:

Ratio analysis	*Unitary analysis*
Given: $\frac{7}{8}$ lb. cost \$2.30	Cost of $\frac{7}{8}$ lb. = \$2.30
Find: Cost of $6\frac{1}{8}$ lb.	Cost of $\frac{1}{8}$ lb. = $\frac{1}{7}$ of \$2.30 = $\frac{2.30}{7}$
Cost of $\frac{7}{8}$ lb. known	
Cost of $\frac{49}{8}$ lb. is 7 × cost of $\frac{7}{8}$ lb.	Cost of $\frac{8}{8} = \frac{\$2.30 \times 8}{7}$
Hence \$2.30 × 7 = \$16.10 = cost of $6\frac{3}{4}$ lb.	Cost of $6\frac{1}{8}$ lb. = $\frac{\$2.30 \times \overset{1}{\cancel{8}} \times \overset{7}{\cancel{49}}}{\underset{1}{\cancel{7}} \quad \underset{1}{\cancel{8}}} = \$16.10$
	or cost of $\frac{1}{8}$ lb. = $\frac{1}{7}$ of \$2.30
	cost of $\frac{49}{8}$ lb. = $\frac{1}{\cancel{7}}$ of $\overset{1}{\cancel{\$2.30}} \times \cancel{49} = \$16.10$ $\quad 7$
Given: Speed, 70 mi. in 2 hr. and 20 min.	In $\frac{7}{3}$ hr., 70 mi.
Find: Distance in 4 hrs. and 40 min.	In $\frac{1}{3}$ hr., $\frac{1}{7}$ of 70 = 10 mi.
	In $\frac{3}{3}$ hr., 10 × 3 = 30 mi.
2 hr. 20 min. = $2\frac{1}{3}$ hr. = $\frac{7}{3}$ hr.	In 4 hr. 40 min. will run 30 mi.
4 hr. 40 min. = $4\frac{2}{3}$ hr. = $\frac{14}{3}$ hr.	$\times 4\frac{2}{3}$
$\frac{14}{3}$ is $2 \times \frac{7}{3}$	$\overset{10}{\cancel{30}} \times \frac{14}{\cancel{3}} = 140$ mi.
∴ train will run 140 mi. in 4 hr. 40 min.	or in $\frac{1}{3}$ hr. will run $\frac{1}{7}$ of 70
	in $\frac{14}{3}$ hr. will run $\frac{1}{7}$ of 70 × 14 = 140 mi.

It is obvious that the ratio method is shorter and shows a deeper grasp of number relations than does the unitary analyses. Children must be taught to examine the quantities in any example or problem and then to seek simple relationships or ratios between them. If, however, one quantity is no obvious multiple of the other, it will be necessary to find the worth of one and then multiply by the necessary number.

C. Analysis by the Algebraic Equation.—It has long been the custom to introduce algebraic solutions in grades where algebra was taught as a separate subject in the course in elementary school mathematics. Formal algebra is gradually being eliminated from the upper grades of the school course and algebraic solutions are now advocated for grades as low as the fifth year. It was concluded in an earlier chapter that literal solutions must be introduced in the interest of better and readier manipulations of problems. Let us apply the algebraic analysis to any type of problem.

A teacher paid \$64.80 for a typewriter after receiving a teacher's discount of 28 per cent. What was the market price of the machine she bought?

Let x denote market price

28% of x or $\frac{28}{100}x$ denotes discount

$x-\frac{28}{100}x$ denotes cost to teacher

\$64.80 is cost to teacher

$$x-\tfrac{28}{100}x=64.80$$

$$100x-28x=6480$$

$$72x=6480$$

$$x=\frac{6480}{72}=90.$$

In this final analysis, this form of solution differs from the unitary analysis only in the use of the symbol x instead of the letters M. P. (market price). The same mode of reasoning is followed. As this type of problem changes

from discount to insurance, the symbol *Pol.* is substituted for the term *policy*. Had the example been one in commission, the abbreviation *Com.* would have been used to designate *commission* throughout. But the symbol x does not make the analysis more literal than *M. P.* or *Pol.* or *Com.* The algebraic analysis often simplifies the solution, gives a uniform analysis, eliminates awkward expressions, and supplants by a consistent and single literal form a variety of names and symbols. A comparison of the usual form of analysis with the algebraic will show its superiority at once.

Goods are to be sold for \$28.50 a gross and a discount of 5% to the trade must be given. What shall I mark the goods?

Usual analysis

M. P. $= \frac{20}{20}$

Dis. $= \frac{1}{20}$ of M. P.

$\frac{19}{20}$ of M. P. = final S. P.

\$28.50 = final S. P.

$\frac{19}{20}$ of M. P = \$28.50

$\frac{1}{20}$ of M. P. $= \frac{1}{19}$ of \$28.50

$\frac{20}{20}$ of M. P. $= \frac{1}{19}$ of \$28.50

$\times 20 =$ \$30. Ans.

This solution is just as literal as the algebraic but more awkward.

Algebraic solution

x denotes M. P.

$x - \frac{1}{20}x$ denotes S. P.

\$28.50 is S. P.

$x - \frac{1}{20}x = 28.50$

$20x - x = 570$

$19x = 570$

$x = \frac{570}{19} = 30$. Ans. is \$30.

What Algebraic Forms Shall Be Taught in Arithmetic?—It is obvious that addition of polynomials, multiplication and division of polynomials by binomials, factoring algebraic forms, and the like need not be taught for increased skill in practical phases of arithmetic. Only three types of equations need be taught to give the child a better tool for working arithmetical problems.

Case 1. Form $x+5=19$, find x.

Method: 1. Introduce through practical problem.

2. Elicit equation form $x+5=19$.

3. Explain by device of a balance.

$$\underline{x+5 \qquad 19}$$
$$\triangle$$

(i) Elicit: taking equal amounts from the two sides of the balance will not disturb the balance.

(ii) Elicit: $x+5-5=19-5$

$$x=19-5 \text{ or } 14.$$

(iii) Elicit: $+5$ on left side of equation becomes -5 on right side.

4. Introduce through a problem the form

$$x-5=22$$

(i) By device of balance elicit:

$$x-5+5=22+5$$
$$x=22+5 \text{ or } 27.$$

(ii) Elicit -5 on left side of equation became $+5$ on right side.

5. Generalization: Law governing transfer of quantities in simple equations.

Case 2. Form $3x=27$.

Method: 1. Introduce through a problem.

2. Reduce to equation form $3x=27$.

3. Use device of balance.

$$\underline{3x \qquad 27}$$
$$\triangle$$

4. Elicit: dividing each side by 3 does not disturb the balance of the two sides.

5. Elicit: $3x \div 3=27 \div 3$

$$x=27 \div 3 \text{ or } \tfrac{27}{3}$$

6. Multiply instances.

7. Formulation of law for finding x in equations like $19x=\$570$.

Case 3. Form $x-\frac{1}{10}x=\$28.50$.

Method: Teach how to clear of fractions by least common denominator. Practice until habit is developed.

These simple equations constitute the most valuable contribution that algebra can make to the literal computations in elementary school pupils' arithmetic. If begun in the fifth year and continued gradually and persistently they increase both speed and accuracy of manipulation.

D. Analysis by Ratio and Proportion.—Not as Frequently Used Today.—Solution of examples and problems by proportion is not used as frequently today as formerly for many reasons: (1) Proportion was discredited by the impossible problems that were usually listed under it in textbooks. Examples of the following type were the rule rather than the exception: "If 42 men build a wall 52 feet long, 20 feet high, 3 feet thick in 27 days, how long will the wall be if 21 men are set to work for 54 days on a wall 15 feet high and 2 feet thick?" The movement for the socialization of arithmetic brought an elimination of these useless puzzles and a consequent limitation in the use of proportion. (2) The algebraic solution now introduced in the middle of the school course offers a satisfactory substitute for proportion in many types of problems. (3) With the popularization of the ratio mode of analysis as opposed to the unitary analysis, proportion was found too cumbersome in many combinations. The truth of this is seen in the following problem:

A freight train running at the rate of 16 miles an hour, takes 72 minutes between two towns. How long will the special milk train take for this trip if it runs at a speed of 48 miles an hour?

By ratio and proportion:

16 miles : 48 miles = ? min. : 72 min.

$$16 \times 72 \div 48 = \frac{\overset{1}{\cancel{16}} \times \overset{24}{\cancel{72}}}{\underset{3}{\cancel{48}}} = 24 \text{ minutes}$$

By ratio analysis:

Speed of 48 miles = 3 times speed of 16 miles.

Time necessary hence $\frac{1}{3}$ of 72 minutes or 24 minutes. For these reasons proportion in arithmetic is now in a transition state; it has lost its former place of importance, but it has not yet crystallized its new function.

Value of Solution by Proportion.—It would cause a decided set-back in the solution of problems to eliminate proportion altogether, for this mode of solution might bring about economy of time and effort as well as greater accuracy. Let us see whether this is true in the following illustrations:

When hay is selling at \$24.00 a ton, what should be paid for 1750 lbs.?

Ratio Analysis: 1750 is $\frac{7}{8}$ of a ton, but most children would not perceive this relation.

Unitary Analysis: Most children solved this example in terms of the cost of 1 lb. of hay.

$$\frac{1}{2000} \text{ of } \$24 \times 1750 = \text{Cost.}$$

Proportion: \$24 : \$? = 2000 : 1750

$$\frac{\overset{7}{\cancel{1750}} \times \overset{3}{\cancel{24}}}{\underset{40}{\underset{8}{\underset{1}{\cancel{2000}}}}} = \$21.$$

If $1\frac{1}{2}$ inches on a map represent a distance of 360 miles, how great a distance is represented by $3\frac{1}{3}$ miles on the same scale?

Ratio Analysis: $3\frac{1}{3}$ is not a multiple of $1\frac{1}{2}$ and no obvious ratio can be seen.

Unitary Analysis: $\frac{5}{8}$ in. represent 360 mi.
$\frac{1}{8}$ in. represents $\frac{1}{5}$ of 360 mi.
$\frac{3}{8}$ in. represents $\frac{3}{5}$ of 360 mi.
$3\frac{1}{2}$ in. represents $\frac{7}{5}$ of 360 mi. $\times \frac{2}{5}$.

Proportion: $1\frac{1}{2}$ in. : $3\frac{1}{2}$ in. = 360 mi. : ? mi.
$\frac{2}{5} \times 360 \div \frac{3}{2}$

$$\frac{\overset{2}{\cancel{25}}}{\underset{4}{\cancel{8}}} \times \overset{30}{\cancel{360}} \times \frac{2}{\cancel{3}} = 750 \text{ mi.}$$

In these two problems proportion involves less analysis and leads more directly to the answer.

How to Develop Ratio and Simple Proportion.—To teach ratio and proportion, it is necessary to give the child a rich concept of the ratio idea. This is best done through use of ratios rather than through definitions and formal explanations. The following procedure may be found helpful:

(1) Concept of Ratio.—Practice should be given in examples of the type, "2 is what part of 4" and children's answers should be tabulated as indicated.

Teacher Asks	Pupil Answers	Teacher Tells
2 is what part of 4?	$\frac{1}{2}$	The ratio of 2 to 4 is $\frac{1}{2}$; this may be written 2 : 4 or $\frac{2}{4}$ or $\frac{1}{2}$
3 is what part of 6?	$\frac{1}{2}$	The ratio of 3 to 6 is $\frac{1}{2}$; this may be written 3 : 6 or $\frac{3}{6}$ or $\frac{1}{2}$
2 is what part of 6?	$\frac{1}{3}$	The ratio of 2 to 6 is $\frac{1}{3}$; this may be written 2 : 6 or $\frac{2}{6}$ or $\frac{1}{3}$
3 is what part of 9?	$\frac{1}{3}$	The ratio of 3 to 9 is $\frac{1}{3}$; this may be written 3 : 9 or $\frac{3}{9}$ or $\frac{1}{3}$
etc.	etc.	etc.

After sufficient practice is given in this initial step, the children fill out the entire line for "6 is what part of 9," "6 is what part of 8," etc. The meaning of the term ratio, "the number expressing the relation of one quantity to another," is now elicited.

(2) Ratio as a Fraction.—The next step emphasizes the fractional equivalent of every ratio. Children write $\frac{3}{5}$, $\frac{4}{7}$, $\frac{9}{12}$, $\frac{9}{15}$ as ratios and 5 : 8, 7 : 16, 9 : 12 as fractions. This step can readily be deduced from the introductory step outlined above.

(3) Introduce the Proportion.—To introduce the proportion, the two ratios, the fractional forms are used.

Teacher gives	Children are required to write
$\frac{4}{6}=\frac{2}{3}$	4 : 6 = 2 : 3
$\frac{9}{15}=\frac{3}{5}$	9 : 15 = 3 : 5

(4) Introduce Mode of Finding Fourth Term, Knowing Other Three.—The teacher gives 3 : 5 = 9 : ? and asks children to give the fourth term. If they hesitate, recall the fractional equivalents $\frac{3}{5}=\frac{9}{?}$; it is evident that the numerator of the first fraction must be multiplied by 3 to obtain 9, the numerator of the second fraction; therefore the 5, the denominator of the first fraction, must be multiplied by 3 also, hence 15 is obtained. Apply to a number of examples and list the work thus:

Teacher gives	Children give	Analysis shows
3 : 5 = 9 : ?	$\frac{3}{5}=\frac{9}{?}$	9÷3 (to obtain 3, the common multiplier) ×5 9÷3×5 or 9×5÷3, the fourth term.
6 : 7 = 12 : ?	$\frac{6}{7}=\frac{12}{?}$	12÷6 (to obtain 2, the common multiplier) ×7 12÷6×7 = 12×7÷6, the fourth term.

Teacher gives	Children give	Analysis shows
4 : 5 = 12 : ?	$\frac{4}{5}=\frac{12}{?}$	12÷4 (to obtain 3, the common multiplier) ×5 12÷4×5 = 12×5÷4, the fourth term.
3 : 4 = 15 : ?	$\frac{3}{4}=\frac{15}{?}$	15÷3 (to obtain 5, the common multiplier) ×4 15÷3×4 = 15×4÷3, the fourth term.

The teacher now introduces the technical terms, *means*, *extremes*, *proportion*. The analysis of each case shows that the product of the extremes equals the product of the means; that the product of the means divided by one extreme gives the other extreme. Apply to practical examples and problems.

(5) Introduce the Type in Which the Unknown Term is One of the Means.—Children who know how to find the fourth term of the proportion 3 : 5 = 15 : ?, may not have enough insight into the number relations to be able to complete the proportion 3 : 5 = ? : 15. This new form is necessary in problems of the type, "If 72 men working 10 hours a day complete a given work, how many men are necessary when the law limits the hours of labor to 8 a day?" Here the proportion would be inverse, 10 hrs. : 8 :: ? : 72, because more men are needed as the working day is reduced. The mode of explanation and analysis is similar to that suggested in step 4, except that the law of proportion (the famous "rule of three") is enriched by the statement "the product of the extremes divided by one mean gives the other mean." Practical problems in inverse and direct proportion are now given to apply these laws.

6. Proportion Involving the Ratio of Different or Unlike Units.—This step is simple, for in ratio of 2 yards to 2 feet it is merely necessary to reduce the quantities to the same

units. The ratio now becomes 2 yards to $\frac{2}{3}$ yard or 6 feet to 2 feet.

7. Compound Proportion.—In dealing with a problem that involves three ratios we have compound proportion. Thus, the preceding problem may be made to read: "If 72 men working 10 hours a day built one branch of a road in 12 days, how many men are necessary to build the second branch of this road, which is just as long and just as wide, if the 8-hour law must now be enforced and the contract time is 18 days?" Compound proportion may well be eliminated from the elementary school curriculum, but where it is demanded by the course of study, a purely deductive method of teaching it should be followed. One of two solutions may be taught, either the compound proportion or the two successive simple proportions.

Compound Proportion:

$$\begin{aligned} &8 \text{ hrs.} : 10 \text{ hrs.} \\ &\qquad = 72 \text{ men} : ? \text{ men} \\ &18 \text{ days} : 12 \text{ days} \\ &\frac{\overset{5}{\cancel{10}} \times \overset{3}{\cancel{12}} \times \overset{4}{\cancel{72}}}{\underset{2}{\cancel{8}} \quad \cancel{18}} = 60 \text{ men} \end{aligned}$$

Successive Simple Proportions:

$$\begin{aligned} &8 \text{ hrs.} : 10 \text{ hrs.} = 72 \text{ men} : ? \text{ men} \\ &\frac{\overset{9}{\cancel{10}} \times \cancel{72}}{\underset{1}{\cancel{8}}} = 90 \\ &18 \text{ days} : 12 \text{ days} = 90 \text{ men} : ? \text{ men} \\ &\frac{\overset{5}{\cancel{12}} \times \cancel{90}}{\cancel{18}} = 60 \text{ men. Ans.} \end{aligned}$$

Cautions in the Analysis of Problems.—We have seen, thus far, that problems may be analyzed by unitary analysis, by ratio method, by algebraic equations and by proportion, each mode involving some degree of cancellation. There are important cautions that must be kept in mind if we are to make analysis of problems effective.

1. Care with Algorisms.—Looseness of mathematical language betrays looseness of thought. It was shown in an earlier chapter that algorisms of the form of $\frac{3}{5}$ = cost, $\frac{1}{5}$ = $\frac{1}{3}$ of cost, etc., must not be permitted. But it is not wise to become too meticulous about form. To condemn the cancellation form, $\frac{3}{5} \times \$5.70 \times \frac{10}{17}$, because there is here implied a process of multiplication in which one multiplier is concrete, is to insist on futile refinements.

2. Verbal Analysis of Problems Must Express the Thought Processes in the Child's Mind.—Verbal analyses are required because they give evidence that the child understands the process, they train in systematic and sustained thought, and they make an appeal to habit by their repetitions. But all these reasons for indulging in verbal analyses are fraught with danger. They assume that thought accompanies the child's words. When the child recites, "If $\frac{3}{5}$ of the money is \$27.90, $\frac{1}{5}$ of the money is $\frac{1}{3}$ of \$27.90 and $\frac{5}{5}$ of the money is $\frac{1}{3}$ of \$27.90×5 or $\frac{5}{5}$ of \$27.90," he may be repeating the teacher's "therefores," "becauses" and "sinces." One is often moved to question whether the reasons heard in an analysis of a problem are the child's own, whether they satisfy him, or whether they are imposed upon him.

3. How to Secure a Thought Basis for Analysis of Problems.—(*a*) When the process is understood, all verbal analysis should cease. At first the problem is solved rationally and by a consciously determined plan. With clear comprehension and increased familiarity with a type of problem, solutions should be freed from accompanying "talk."

(*b*) Verbal analysis should be given only for new problems and for new parts or new elements in problems.

(*c*) The verbal expressions in the analysis of problems should, as far as possible, be the children's own and not the scientifically accurate language of the textbook or of the teacher.

(*d*) Wherever possible the analysis of a problem should be given not verbally, but by diagram. In many of the problems cited in this chapter, a diagram correctly made by a child would give indisputable evidence of clearness of comprehension and of a thorough grasp of the mathematical relations.

(*e*) All verbal analysis should be eliminated from the first four years. In these early grades verbal analysis degenerates into twaddle, into a mere mouthing of words utterly devoid of thought.

3. Solving Problems by Rules or Formulæ.—In the tabulation of methods of solving problems we find (1) the graphic method, (2) the analysis with its four distinct types, and (3) the use of a rule or formulæ. The development of the inductive method brought the use of rules into bad repute. Teachers had so long given lessons that began with general laws and formulæ, had presented symbols before thought—symbols which therefore fail to symbolize—that a complete reaction to solution by rules followed. If the rule or the formula comes at the proper time and has a rich foundation of experience and comprehension, it has many values. It gives a mastery of necessary symbols, it makes a general branch of knowledge permanent, it aids in reducing necessary facts to habit, it makes long verbal analyses unnecessary, and it gives power to deal with abstractions. Let us assume that children have solved the following type by rational analysis: "If I paid \$5.40 for goods after receiving a discount of 20%, then $\frac{4}{5}$ of the marked price is \$5.40, $\frac{1}{5}$ of the marked price is $\frac{1}{4}$ of \$5.40 and $\frac{5}{5}$ of

the marked price is $\frac{1}{4}$ of \$5.40×5 or $\frac{5}{4}$ of \$5.40." Then they may be shown the short cut, "$\frac{5}{4}$ of \$5.40," and thereafter they may find $\frac{5}{5}$ or 100 per cent by dividing the known number by the known fraction. Similarly, in finding the cost of one yard if $1\frac{3}{4}$ yards cost \$2.73, it is not necessary to perpetuate the form, "If $1\frac{3}{4}$ yards cost \$2.73, $\frac{1}{4}$ of a yard costs $\frac{1}{7}$ of \$2.73 and $\frac{4}{4}$ of a yard cost $\frac{1}{7}$ of \$2.73×4 or $\frac{4}{7}$ of \$2.73." When the teacher has every guaranty that the children understand the analysis, she should lead them to conclude, "To find the cost of one, knowing the cost of any quantity, divide the cost by the quantity." In the very nature of the case, rules or formulæ will come at the end of the term's work and toward the end of the school course, when they summarize the experience and the insight resulting from the long use of the full form of analysis and rationalization.

IV. Checking the Answer

It was shown that there are four steps which a child should consciously take in solving any problem, I. Grasping the Problem, II. Planning the Solution, III. Executing the Plan and IV. Checking the Answer. The discussion thus far concerned itself with the first three of these steps. The fourth need not detain us long, not because it is not of coördinate importance with the others in making for accuracy of results and in developing a helpful mode of thought, but because it was already treated in Chapter IX. There we stopped to emphasize the importance of checking answers, and of inculcating a critical attitude toward one's own work. We also set forth ways of testing results whether by accurate proof or by mere approximation.

CHAPTER XIV

DENOMINATE NUMBERS AND MENSURATION

TABLE OF WEIGHTS AND MEASURES

Place of Denominate Numbers in an Elementary Course.—The importance of the study of units of measure in arithmetic is explained by the need that we have for them as media of exchange and of evaluation in social life. The character of these units of measure is often an index of the degree of development of a people. These measuring units have kept pace, step by step, in their progressive changes, with the growth in commerce, industry and science. The cruder the units of measure, the more primitive are the people who use them; the more interrelated and specialized a society becomes, the more refined and accurate are its units of measure. But a clear perception of the importance of denominate numbers is no justification for teaching much of what is traditionally included in textbooks and courses of study. The final criteria of what to teach in denominate numbers are the same as those that obtain in all studies and in all branches of arithmetic, the needs and the practices of business life. Because of the extent of the field in denominate numbers, discriminating selection and liberal elimination are essential.

Gradation in the Use of Measurements.—There are at least three important steps that mark a graded teaching series which develops rich concepts of units of measure. The first consists of exercises of *general comparison.* Objects, lengths, and weights should be compared. Two textbooks held on the palm of one hand may be compared with a

number of books balanced in the other. Lines of varying length are drawn on the board and their relative lengths are judged. The distances between any two objects in the room may be similarly compared with other distances. So, too, solids, boxes, and rectangles may be used in these "judging" exercises to train the senses and give a feeling for *more, less, larger, smaller, heavier, taller,* etc. In the beginning the elements contrasted are large and the differences are very marked. As progress is made both the size of the objects and the differences in distance, bulk, and weight are gradually decreased.

In the second step in the teaching of measurements defined but varying units are used. A cup, a chalk box, a step, a span of the hand, a drinking glass may be the measuring agent. The use of *fixed* and legally *defined units* marks the third step. The foot, quart, inch, dollar, cent, ounce, pound, or square inch are examples of these fixed units. These three—the undefined unit, the defined and varying unit, and the fixed and unvarying unit mark the sequence to be followed in teaching denominate numbers.

Omit All Tables Not Commonly Used or Those Necessary Only in Technical Industries.—The elementary school course seeks to develop general skill in arithmetic which can be applied to any specific condition in which an arithmetical situation arises. For this reason, it must limit itself to tables of weights and measures that are used by all citizens in the course of daily relations. To teach troy weight or apothecaries' weight is to force all children to learn what a very few will need. Surveyor's measure, long ton and hundredweight, cord-foot, cord, perch and stone measure must be put into the same category of measures to be omitted. Children who learn liquid measure or the linear tables are acquiring units used by all citizens in the routine of life. This standard of selection would lead us to teach only the following in denominate numbers: 1, Tables: linear, square,

dry, liquid, avoirdupois, federal money, time and metric system; 2, such isolated but useful concepts as dozen, score, ream, gross, 16 fluid oz. in 1 pint, etc.; 3, the names and equivalents of the most common foreign monetary units, e.g., pound sterling, shilling, franc, lira, mark, and rouble. An examination of commonly used textbooks shows at once the complete absence of any social standard of selection in denominate numbers.

All Units Must be Taught through Use.—While it is true, psychologically, that number is not a sense fact, we must ever be mindful of the fact that the number idea has its origin in sense experience. Children should learn foot, inch, yard, pound, pint, gallon, by using these units and measuring with them in a variety of situations. To most children *foot* recalls the equivalent twelve inches, but the association is purely verbal. None of these units calls up the measure it represents. Teachers know how few children who have used these common units for at least three years can give approximately the height of the door, the width of the window, the weight of the books carried to and from school, or the length and width of a surface containing 120 square inches. Children in a 6 B class, when asked the height of an ordinary room, gave answers varying from 25 to 60 feet. Only about 10 per cent gave the correct answer. When a class of college juniors was asked the height of an ordinary four-story brick building, the answers ranged from 35 to 200 feet; not more than 20 per cent gave a reasonable estimate. Too often units of measure are words and remain empty ideas because of early teaching which neglected to give a proper perceptual setting.

How to Teach a Unit of Measure.—In teaching a unit of measure, e.g., the pound, *use* and *sense appeal* must be the keynote. A balance-scale, a number of weights, and a spring scale must constitute part of the school equipment. Children are asked to hold a pound weight in each hand; to hold a

pound weight and then replace it with a number of books judged to weigh a pound; to estimate the weight of various boxes containing sand and of vessels containing water, etc. In each case the estimate is checked up. These exercises should be repeated for two or three minutes each day until the pound means a definite weight and recalls a certain resistance. In similar exercises the quart, the gallon, the pint, the foot, the ounce must be taught. Children who spend two minutes each day judging various distances in the class and checking their answers will, at the end of a fortnight, show marked improvement in estimations of distance. Children should know the width of an average city lot, the height of a room, of an ordinary building, and of a modern "skyscraper," so that 20 feet or 100 feet will call up a distance twice the height of an average room or the depth of a city lot, respectively. "Two hundred and thirty-one cubic inches" should be learned by making a cardboard box 7 by 11 by 3 inches, filling it with sand, and then comparing the contents of the box and of a gallon. Unless units are learned through use, the tables based on them are associations that can never function intelligently.

A Knowledge of a Variety of Important Units Should Precede the Formulation of Tables.—The complete tables, e.g., 12 inches = 1 foot, 3 feet = 1 yard, $5\frac{1}{2}$ yards, $16\frac{1}{2}$ feet = 1 rod, 320 rods = 1 mile, etc., should come late in the school course and should be preceded by a knowledge of common units taken from all useful tables. The child should know quart, cent, dime, foot, week, before knowing any one table in its usual formulation. From each table we select such units as are most frequently used and such as will lend themselves most readily to objective demonstration. For these reasons children learn foot before inch, quart before pint, cent before dollar, etc. The New York City course of study follows this organization and a general skeleton of its

Grade	Federal Money	Linear Meas.	Liquid Meas.	Time	Misc. Units	Avoir. Weights	Dry Meas.	Square Meas.	Cubic Meas.	Metric System	Foreign Money
1 A	Undefin	ed units									
1 B	c.-d-5c.,	ft.	quart								
2 A	25c., 50c., 1.00	in.	pint								
2 B		yard	gallon	hr., day, wk.-tell-ing time							
3 A				year	doz.	lb.					
3 B	Problems	using the	above				qt.,pk.,bu.				
4 A		mi. in ft.				oz.					
4 B								sq. in., ft., yard			
5A–6A	Complete	Complete	Complete	Complete	Comput-ing per gross, 100, 1000, etc.	Complete	Complete	Complete	and use a	bove in pr	ob.
6 B									all and apply		
7A–8B	Problems	using abo	ve and							all useful parts	£., s., fr., rouble, etc.

assignment in denominate numbers is, therefore, submitted by way of illustration.

The Tables to be Formulated by the Children.—In teaching a table it is not advisable to present it in complete form either on a large cardboard or on a page in a book. Children who have been taught the units of any measure should be led to discover the necessary equivalents, and should formulate these relations, step by step, as they progress. Such a mode of developing a table gives it greater meaning and an attraction possessed only by those things that are the product of our own efforts. The old mode of teaching followed the archaic order, words, thoughts, and things; the modern procedure—things, thoughts, words—is not only more psychological but affords, among other teaching advantages, opportunities for rational rather than mechanical presentation, multiple sense appeal, reduced amount of drill, greater interest and self-activity, and a thorough perceptual basis of number facts.

Denominate Numbers to be Used Only in Real Situations.—The complaint that school arithmetic is too often divorced from business practice is found to be justified when one investigates the types of problems given in denominate numbers. Impractical and impossible examples fill pages of standard texts and take up weeks of classwork all to no avail. True, these exercises develop skill in manipulation, but not in a form of work that ever arises in real life. The following examples are taken from books much used in school systems:

Reduce 2 A., 4 sq. rods, 8 sq. ft. to sq. in.
Reduce 247,856 inches to higher denominations.
Reduce 1 mile, 1 rod, 1 yard, 1 foot, 1 inch to inches.
Express 58 rods, 2 yards, 2 feet, 2 inches as a decimal part of 3 miles.

Each of these examples is an artificial situation evolved by textbook writer or teacher, and is not even remotely related to the practical needs of life.

It is usual to find children computing a quantity determined by troy measure in terms of units of avoirdupois. A recent book asks: "A silver bowl weighs 3 lbs. avoirdupois; what did it cost at $1.00 per troy ounce?" Of course the situation is not true to life. A principal was upbraiding one of his teachers who permitted children to solve the example by reducing 3 lb. to 48 ounces avoirdupois and then to 4 lb. troy. But it was new to the principal that a grain troy is not the same as a grain avoirdupois. Not only is the problem artificial, but its solution is complicated. Problems of the type of "if a man buys 5½ bu. chestnuts at $2.00 per bu. and sells them at the rate of 10¢ per liquid quart, what is his profit?" must also be condemned, for such knowledge is useful only to those bent on dishonest practice.

Correlate Denominate Numbers with other Branches of Arithmetic. Emphasize Oral Work.—As far as possible, denominate numbers should not be taught as an independent branch of arithmetic. Units of measure must be incorporated in work in fractions, in decimals and in fundamental operations of whole numbers. Let us consider the following questions: "How many gallons in a barrel containing 44 quarts?" "In 1½ ft. how many inches?" "In ¾ of a yard how many feet?" "In 8½ ft. how many inches?" "A length of 5 in. is what part of 1¼ ft.?" It is evident that these examples give practice in denominate numbers and should be oral and rapid to train children in the form in which these examples occur in actual business.

Teaching the Arithmetical Operations in Denominate Numbers.—The essential operations in denominate numbers are: (1) the four fundamental operations; and (2) reductions to lower and higher forms. Little need be said about the former. Few teaching difficulties arise. Not much attention should be devoted to addition, subtraction, multiplication, or division of denominate numbers, for they are

of unusual occurrence in the average person's work. If these topics must be taught, a deductive method will be found advisable.

In teaching reduction ascending or descending, it is well to begin with a practical problem so that the children may see the importance of the process. If this work is graded carefully, preceded by plenty of oral exercises, and confined to reductions that do not include more than three successive ascending or descending units, little or no difficulty is experienced. The gradation of the most important steps is summed up below:

1. *Reduction descending.*

(*a*) One step; 7 yards to feet, 5 rods to yards, 16 hours to minutes, etc.

(*b*) Two steps; 7 yards to inches, 5 rods to feet, two days to minutes, etc.

(*c*) One step and a lower denomination; 5 yards, 1 ft., to feet; 2 hours, 15 minutes, to minutes.

(*d*) Two steps and a lower denomination; 5 yards, 1 foot, to inches, 3 hours, 30 minutes, to seconds.

(*e*) Three steps, 1. Without lower denominations.
2. With lower denominations.

(*f*) Fractional reductions; $\frac{3}{4}$ mile to yards.

2. *Reduction ascending.*

(*a*) One step, even reduction; 24 inches to feet, 360 seconds to minutes.

(*b*) Two steps, even reductions; 108 inches to feet and yards.

(*c*) One step, remainder after reduction; 37 inches to feet, gives 3 feet 1 inch.

(*d*) Two steps, 2 remainders after reduction;

12 | 122 inches
3 | 10 feet and 2 in.
3 yd. and 1 foot

(*e*) Three steps, 1. Even reduction.
2. With 2 or 3 remainders.

3. *Problems which embody the following operations.*

(*a*) $\frac{1}{3}$ of 2 yards and 2 feet is ?

(*b*) 2 yards, 2 feet are what part of 3 yards 1 foot ?

(*c*) 3 yards, 1 foot $\times$ 8 ?

(*d*) A barrel contains 24 gallons, 2 quarts. How many bottles each holding 3 pints can be filled, etc.?

The Metric System.—*Why It Should Be Taught.*—The metric system must be made part of the course in denominate numbers because: (1) Its decimal characteristics make most of its multiplications and divisions a mere matter of moving the decimal point; (2) it is so common in Europe that foreign bills and commercial paper sent to America are computed in terms of its units; (3) it is the system employed, almost exclusively in science; (4) the movement for its introduction into the United States is gaining popularity. While there are no immediate prospects that the metric system will be legalized here, it is necessary that the school graduate understand what he sees about this system in the newspapers. Although the scientific superiority of the metric system is conceded and its uniformity and simplicity of manipulation are admitted advantages, there is a strong sentiment against abandoning our clumsy measures: first, because of the binding force of tradition, and second, because of the necessity of completely changing all our tools, machines, computations, etc., from the one basis to the other. To many, the cost and the very real inconveniences involved in such a change seem to be great obstacles.

Teaching the Metric System.—*Teach the Three Basic Units.*—The first step is to teach children, as objectively as any other units, the three basic measures of the metric system: The meter, 39+ inches; the liter, the contents of cube the side of which is .1 of a meter; the gram, the weight of a cubic millimeter of distilled water.

Teach the Diminutive Prefixes.—After the children have learned the metric units of length, capacity and weight,

have a clear image of their equivalents, and can estimate distance, capacity, and weight, in terms of meter, liter, and gram, the diminutive prefixes should be evolved inductively by using our decimal system as an aid. The .1 gives the *deci*, the .01 gives the *centi* and the .001 gives the *milli*. These prefixes will readily lead to .1 meter or decimeter, .01 meter or centimeter, .001 meter or millimeter, etc.

Use This Part of the Metric System in the Arithmetical Operations Characteristic of Denominate Numbers.—Examples in finding areas, volumes, equivalents in American measures, etc., should now be given so that the children master the new system, see its use, learn the abbreviations, and are practiced in recognizing equivalents.

Teach the Augmentative Prefixes.—The prefixes of increase are more difficult for children because they are Greek derivatives. They must be taught deductively and with due emphasis on the American equivalents. We must make sure that the *deka* ($\times 10$), *hekto* ($\times 100$), *kilo* ($\times 1000$), and *myria* ($\times 10000$), are thoroughly understood. The augmented measures are now formulated and applied to a variety of examples as suggested above.

The Comparative Chart.—All work in the metric system should be given with the comparative chart in full view. These charts showing our measures and the metric equivalents can be obtained from the appropriate Federal Bureau and from many manufacturers of school equipment. At every stage of the work we must give children this sense of equivalents. Unless we constantly translate one measure into the other, the metric units will degenerate into mere names.

When to Teach the Metric System.—It seems desirable to postpone the metric system until the seventh school year for many reasons: (1) We must wait until the work in elementary science offers a motive for learning the new system. 2. Unless decimal manipulations are reduced to habit the metric forms will cause trouble. 3. Only when the child

has complete control of our own units of measure and the arithmetical manipulations of them,—only then will the new operations have meaning.

MENSURATION

Why It Should Be Taught in the Elementary School.—The topic of mensuration is subject to the same teaching abuses as that of denominate numbers. This branch of arithmetic is expanded beyond all rational bounds, more because the field is rich in mathematical possibilities than because of any pedagogical reason. It is necessary to formulate reasons that justify the teaching of mensuration so that they may serve as a standard in terms of which material may be selected and eliminated.

Mensuration must be taught: (1) because of its variety of practical applications in computing lengths, distances and perimeters, in drawing to scale, in finding areas, in ascertaining contents of bins and tanks, in computing the cost of digging cellars, paving streets, laying sidewalks, papering walls, etc. (2) Work in mensuration is a means of employing the items of the various tables of measure; it constantly reviews their units and their relations. (3) A body of useful terms is acquired. In mensuration the child acquires clear concepts of distance, area, volume, parallel, geometric forms, acute, obtuse, perpendicular, altitude, etc.—a host of ideas found in many subjects and in a variety of practical activities. But mensuration must never be taught in the elementary school because it paves the way for the study of geometry or because it introduces the child to geometric deductive reasoning We may indulge in no such disciplinary aims until the business world is thoroughly satisfied with the arithmetical accuracy and skill of the school graduate.

Teaching Principles Governing Mensuration.—1. *Knowl-*

edge the Result of Actual Experience.—To insure clear comprehension, children should be required to perform all measurements, drawings, and constructions involved in teaching any principle of mensuration. Every child must be led to formulate the law of areas of rectangles, triangles, parallelograms, cylinders, etc. By a process of construction and superposition of parts, this inductive, heuristic method is possible beyond all question. No rules should be given nor should the teacher's blackboard drawings be the only objective aid. Every arithmetic lesson in mensuration should also be a lesson in drawing and construction work for each pupil. The lessons that conclude this chapter will illustrate this principle.

2. *No Geometric Demonstration.*—Such rules of mensuration as require elaborate geometric demonstration and purely abstract reasoning should be reserved for the high-school course in mathematics. The volume of a sphere, of a cone, of a frustum of cone or pyramid, the area of trapezoid and trapezium, of the surface of a sphere or cone are illustrations of the point in question. To include these in arithmetic is to misconstrue the aim of the course and the purpose of elementary school education.

3. *The Mode of Computation Must be the One Used in Industry.*—To continue teaching children the logical practices in papering, plastering, and carpeting, evolved by teachers and authors of books in arithmetic, but not used in actual industry is to widen the breach between classroom and life. Children are required to find the cost of papering a room $20 \times 15 \times 9\frac{1}{2}$ ft. at $12\frac{1}{2}$c. a sq. yd., deducting for two windows, each 6×4, 1 door, $4 \times 6\frac{1}{2}$ ft., and a baseboard 1 ft. high. To find the answer the total number of square yards in the walls is found, the area of unplastered parts is deducted, and then the cost is computed. But in actual life the number of rolls necessary is found first. The paperhanger charges (1) for the paper at — dollars per roll, and (2) for

labor at — dollars per day per man. The elaborate procedures found in school-books which deduct one-half of the openings because of the difficulty of turning corners, or which indulge in imaginary business practices, result in misinformation and constitute a serious source of waste.

4. *Practice in Oral Estimates.*—Much of the work in mensuration should consist of exercises which call for approximate lengths, areas, volumes, perimeters and costs. This work, done orally and continued until a fair degree of accuracy is developed, is exceedingly valuable for practical needs in various business situations.

5. *Insist on Short Cuts in All Manipulations.*—Mensuration entails no little multiplication and division of numbers. It is desirable to incorporate in this branch of arithmetic as many short cuts as possible. In inventing problems the numbers should be deliberately chosen so as to encourage the use of short methods. Instead of finding the cost of excavating a cellar $38\times15\times8$ yards, select one that is $12\frac{1}{2}\times39\times8$ yards. The children naturally multiply $12\frac{1}{2}$ by 39 and then multiply the result by 8. Show them the need of looking for short cuts before beginning the mechanical operations. In this example we have $12\frac{1}{2}\times39\times8$, which is $12\frac{1}{2}\times8\times39$; $12\frac{1}{2}\times8=100$; $100\times39=3900$. If possibilities for short cuts are deliberately introduced we can develop in children the habit of looking for them.

6. *The Metric Units.*—Just as soon as the metric system is mastered, it should be applied in examples in mensuration. The areas of rectangles and triangles, the volume of vessels, etc., can be found in metric units as well as in our present units of measure.

ILLUSTRATIVE LESSONS IN MENSURATION

Drawing to Scale.—We must stop to illustrate some important topics in mensuration. Drawing to scale is a use-

ful application of facts of denominate numbers. The steps that may be followed in developing the idea of scale are suggested below:

1. Draw on board, cover of chalk box, circular cover of inkwell, top view of book. Use exact measurements. Children do same on board or on paper.
2. Introduce new need, draw floor on board. Elicit, "We can draw a figure having the shape but not the size of the floor."
3. Introduce idea of scale. Draw plan of floor; scale, "inch on board represents one foot of the floor measurement."
4. Draw to scale a wall, the hallway, the street, the playground.
5. Introduce varying scales for same object. Draw plan of the floor to scale $\frac{1}{4}$ in. = 1 foot, 1 inch = 1 foot, 1 inch = 1 yard.
6. Draw three rectangles on the board, each 4×3 feet. In one rectangle draw plan of the room, scale 3 inches = 1 foot; in second plan of yard, scale 1 inch = 1 foot; in the third, a plan of the street, 1 inch = 2 feet. Children often say that Germany and the United States are the same size or that South America is half the size of the United States. Upon investigation we find a full-page map of Germany and a full-page map of the United States, or a full-page map of South America and a double-page map of the United States. The changing scales are not taken into account and the child makes the errors quoted above. These exercises in drawing to a scale give valuable aid in interpreting distance and size in geography.

Area of Rectangle:

1. Practice in finding area of cover of cigar box, of cover of book, etc., using a square inch and actually making measurements.
2. Practice in estimating similar areas.
3. Lead children to perceive need of finding a means of computing large areas other than by actually measuring the surface by means of the square inch or the square foot.
4. Children and teacher draw rectangles 2×3, 3×4, 4×5 inches or feet, on papers or on blackboard. Develop law by analytical drawings like the following:

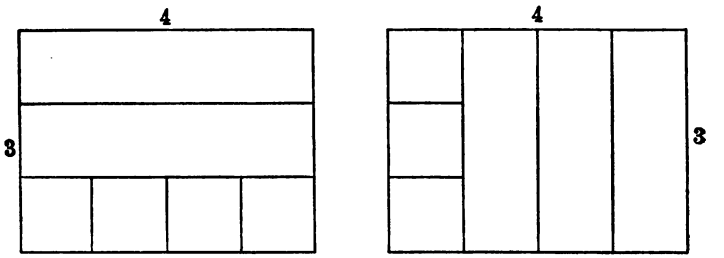


FIG. 22.

Elicit, "4 sq. in. in one row, 4 sq. in. $\times 3$ (in 3 rows), etc." After a few rectangles are so treated, lead class to formulate the law, "the number of square units in one row multiplied by the number of rows," 4 sq. in. $\times 3 = 12$ sq. in. or 3 sq. in. $\times 4 = 12$ sq. in. Avoid 4 in. $\times 3$ in. $= 12$ sq. in., $4 \times 3 \times 1$ sq. in. $= 12$ sq. in., because both forms are mathematically untrue.

5. Grade examples in area: (1) length and width given in same denominations; (2) length given in one denomination, e.g., ft., width in another, e.g., yards; (3) fractions introduced $36\frac{1}{2}$ ft. by $12\frac{3}{4}$; (4) finding area when length and width are not given directly, e.g., find the area of a floor 30 ft. long and half as wide; find area of a square floor 16 ft. on a side. These examples, although hypothetical, are permissible because they are interesting to children and appeal to the puzzle instinct. They give perfect control of this basic principle of mensuration.
6. Introduce formula, base $\times$ altitude, as area of any rectangle.

Area of Parallelogram:

1. Be sure children know distinguishing characteristics of the parallelogram. Let them draw a rectangle and a parallelogram and point out all differences.
2. Give problem showing need of finding area of parallelogram.
3. The following procedure is suggested: all constructions to be made by children.

The parallelogram is drawn. The two triangles *a* and *A* are formed by drawing vertical lines to base. Cut out triangles *a* and superimpose on *A*. Elicit: (1) the two triangles

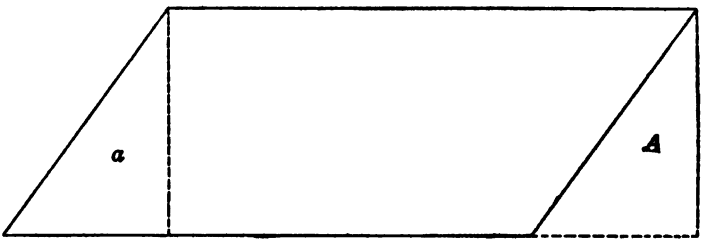
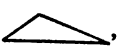
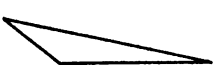
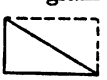
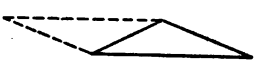
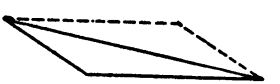


FIG. 23.

are equal in area; (2) the new figure (when *a* is on *A*) is a rectangle; (3) the area of the rectangle is the same as the area of the parallelogram. Lead children to conclude that the area of parallelogram = base × altitude. *Caution:* Be sure that the children do not confuse the dotted line representing the altitude with one of the parallel sides of the parallelogram. Solve practical problems.

Area of Triangles:

1. Practical problems showing need of knowing how to compute area of triangular surfaces.
2. Children draw a rectangle, a variety of parallelograms, and then one diagonal in each. The resulting triangles are now cut out and superimposed. Elicit: the area of a triangle is half the area of a parallelogram or rectangle.
3. Children and teacher draw a right-angled triangle, triangle having the shape of  , and another  . Complete the rectangle or the parallelogram as follows:

Triangles are now cut out and the same conclusion is elicited, viz., the area of a triangle is half the area of a rectangle or parallelogram.

4. Lead children to formulate the law: area of a triangle is half the product of base by altitude. Caution: Be sure children know the meaning of altitude.
5. Practical problems are now solved.

Area of Trapezoid.—The place of the trapezoid in an elementary school course may well be doubted. The author would counsel its omission, but since so many courses of study make it mandatory, the following method is suggested:

1. Give problem involving the area of a trapezoid. Lead children to observe its distinguishing characteristics.
2. Children and teacher draw the trapezoid. Extend the parallel sides a and b, lay off the length of a on the dotted extension of b. Lay off the length of b on the dotted extension

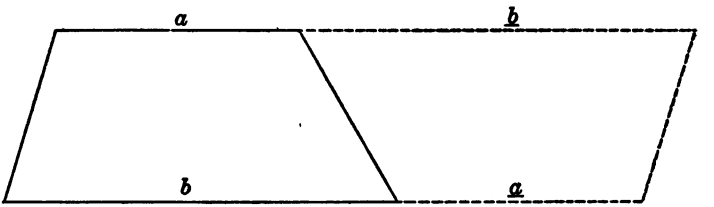
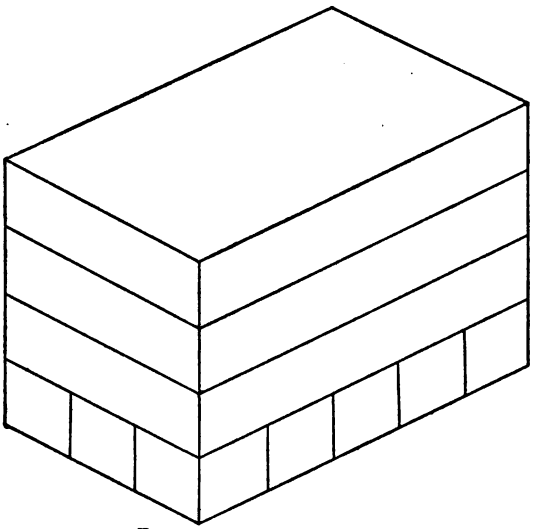


FIG. 24.

of a. Connect the two lines. Lead children to note that: (1) the entire figure is a parallelogram; (2) the large parallelogram is made up of two trapezoids.

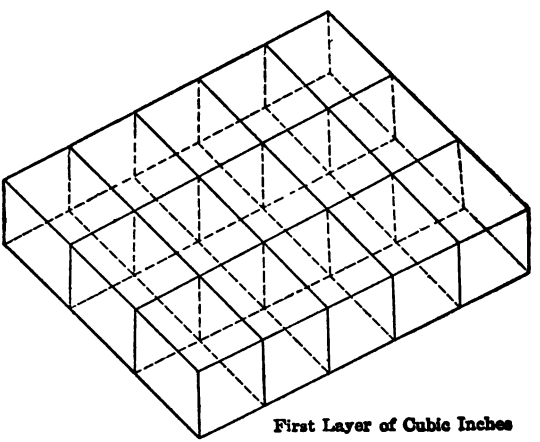
3. Children cut out the two trapezoids and superimpose one on the other. Elicit: (1) the two trapezoids are equal in area; (2) each trapezoid is $\frac{1}{2}$ the large parallelogram, hence,
   area of trapezoid $= \frac{1}{2}$ area of large parallelogram.
   But area of a parallelogram is base $\times$ altitude, hence,
   area of trapezoid $= \frac{1}{2}$ (base $\times$ altitude).
   But the base of the parallelogram is the sum of the two parallel sides of the original trapezoid, hence,
   area of trapezoid $= \frac{1}{2}$ (sum of two parallel sides $\times$ altitude).
4. Practical Problems.

Volume of Rectangular Solids.—The plan suggested for teaching the area of a rectangle can be followed for teaching



Box 5 x 4 x 3

FIG. 25*a*.



First Layer of Cubic Inches

FIG. 25*b*.

the volume of rectangular solids. At first, contents of cigar boxes and the like are measured with a cubic inch and then practice is given in estimates. In developing the law we must avoid, "3 in.×5 in.×4 in.=60 cubic inches." The law is derived by finding that there are 15 cu. in. in the first layer of cubic inches and that there are four layers of them. To teach this law concretely blocks are necessary which are laid in four layers of 15 each. After the law is developed, thoroughly understood, and applied to a number of examples and problems, the formula, "length×width ×height," or L×W×H, is given as the short form.

The Circle.—There are no teaching problems in leading children to understand circle, circumference, radius, diameter, "$3\frac{1}{7}$ times the diameter," etc. All these facts and relations can readily be demonstrated objectively. Every child can be led to discover that the circumference is $3\frac{1}{7}$ times the diameter. A hoop is rolled on a straight line on the floor, starting with the point X and ending when the X once more touches the floor. The distance traversed by the hoop is compared with the distance across its widest part, and the resulting ratio of $3\frac{1}{7}$ is easily obtained.

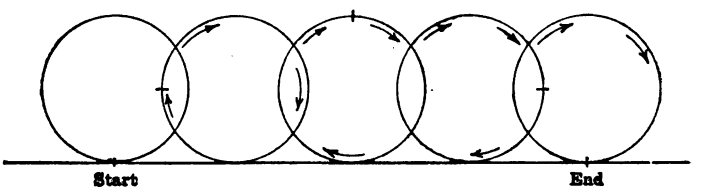


FIG. 26.

The area of the circle can be taught deductively or from the class text. The children thus obtain the law πr^2 and apply it forthwith. But pedagogical authorities, for some reason, eschew such economy in teaching a topic of doubtful worth to the average school graduate. The following procedure is, therefore, outlined for the teacher whose

course of study prescribes the area of the circle and whose supervisor insists on an inductive development.

After the area of the circle has been properly motivated, children and teacher draw a circle and cut it up into sixteenths by means of diameters. Each child then cuts

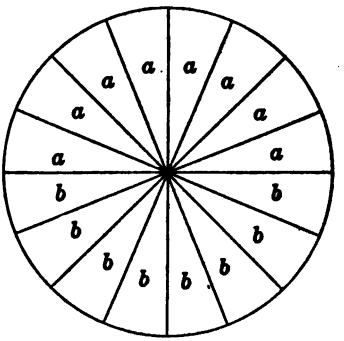


FIG. 27.

across the horizontal diameter and now has two semicircles, each showing 8 sectors thus (Fig. 28):

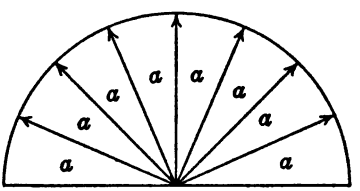


FIG. 28.

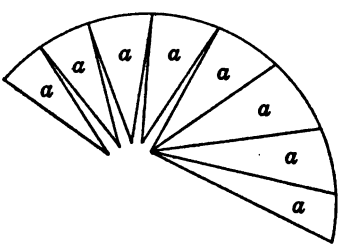


FIG. 29.

Beginning at the center, cut along each radius to a point almost touching the circumference. The first and the last divisions are now taken up by the forefinger and the thumb (Fig. 29) and gently stretched until the sectors of the semicircle are arranged in a horizontal row (Fig. 30).

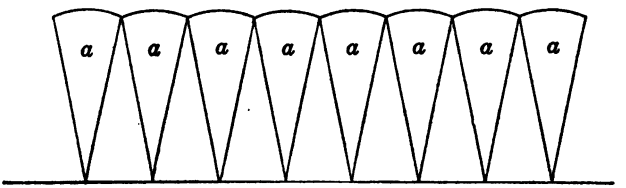


FIG. 30.

The same constructive operation is performed with the lower semicircle and the two sets of sectors are dovetailed into each other to form a figure closely approximating a parallelogram.

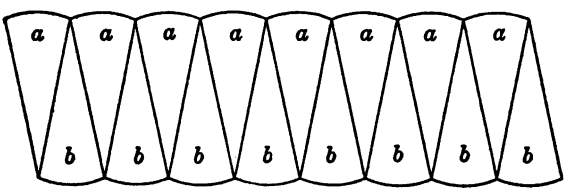


FIG. 31.

The following equations are then elicited from the class:

area of parallelogram = area of complete circle,
area of parallelogram = base × altitude,
area of the circle = base × altitude.

But base is $\frac{1}{2}$ of the circumference and the altitude is the radius, hence,

area of circle = $\frac{1}{2}$ circumference × r.

Substitute for circumference $3\frac{1}{7}$ × diameter or πd, hence

area of circle = $\frac{1}{2}\ \pi d \times r$.

Substitute, " 2 radii " for diameter and the equation becomes,

$$\text{area of circle} = \tfrac{1}{2}\ \pi 2r \times r.$$

Cancel,

$$\text{area of circle} = \pi r \times r = \pi r^2.$$

Other Areas and Volumes.—These illustrative lessons emphasize the need of leading children to discover the laws of mensuration through drawing and construction work, and also of beginning each lesson with a practical situation as well as with a clear comprehension of the characteristics of the geometric form to be taught. The law for the area of the surface of a cylinder, the law for the cubical contents of a pyramid or cylinder, or any other law that the course of study may demand in mensuration can be taught by methods which embody the aim, the organization, and the general characteristics of the illustrative lessons given in this chapter.

CHAPTER XV

BUSINESS FORMS

Place in the Course of Study.—Business forms have an undisputed place in the course of study because of their importance in all business relations. Heretofore, courses of study have made them an incidental rather than a major element in the teaching of arithmetic. It was erroneously conceived that most business forms have little disciplinary value, and were therefore given only passing attention. Every business form motivates itself because it appeals to the child as necessary in his present life as well as in his later work. For this reason it secures maximum attention and concentration from pupils and leads them to perform the work with greater care than is bestowed upon a topic that does not seem so useful.

What Business Forms Shall be Taught in the Elementary Course?—Every necessary and commonly used business form must be taught the child before the elementary course is completed. The business forms that merit a place in the course of study in arithmetic are given below in the order in which they may be taken up.

1. Bills.
   - (*a*) For articles purchased on same day.
   - (*b*) For articles purchased on a different days.
   - (*c*) Bills receipted in full payment.
   - (*d*) Bills showing payment on account.
   - (*e*) Bills for labor or services rendered.
   - (*f*) Bills showing a discount or a double discount.

(*g*) Bills showing credit items allowed, i.e., a person is charged $25.40 for 6 articles purchased but allowance is made because 2 of these, costing $6.10, were returned.

2. Invoice.

Children must be led to understand that an invoice is a memorandum of goods shipped, showing quantity, quality or style, cost, mode of shipment, etc., so that the purchaser may be able to check up what he received with what he will be charged. A bill, on the other hand, gives most of this information, but comes in the nature of a request for payment.

3. Monthly statement.

4. Orders.

In teaching children to order goods, catalogs should be used and the necessary information to be sent must be decided on before the order is written. Children can be led to see the need of telling (*a*) the exact date the article is ordered, (*b*) its catalog number, (*c*) quantity and quality such as color or size, etc., (*d*) date of shipment, (*e*) mode of shipment, (*f*) place to which goods are to be sent, and (*g*) mode of payment. It is not always necessary to give all this information, for items (*d*) and (*e*) may be of no importance in some forms of business; item (*f*) must be given when goods are sent to warehouses or to a customer of the firm making the order, and item (*g*) applies only in limited cases.

5. Receipts.

In making out receipts vary the conditions, i.e., receipt in full payment, on account, for goods received as well as for money. In all cases children should make out the stubs as well as the receipts.

6. Checks.

In connection with the study of checks, the children should be taught: (*a*) making out stubs; (*b*) paying to the order of a firm, or to the order of John Jones,

Treasurer, not John Jones personally; (*c*) paying to order of cash; (*d*) indorsements; (*e*) certified checks; (*f*) how a check serves as a receipt.

7. Deposit slip.

   Fill out slip showing bills, specie and checks ready for deposit in a business bank.

8. Cash account.

   Page from cash book of any retail dealer.

9. Household account.

   Principle of cash account applied to household receipts and disbursements.

10. Personal account or ledger account.

    The term "personal account" may refer either to (*a*) an account kept by one person to show his private receipts and disbursements, or (*b*) to a ledger account showing the purchase and payment status of any one customer. In business the usual significance of "personal account" is the latter. These two forms of personal account are illustrated below:

PERSONAL ACCOUNT OF JAMES COLLIN			DR.		CR.	
1916						
Jan.	3	Cash on hand	2	45		
	3	For Skates			1	15
		For Cap and Ankle Supports			1	05
	5	Helping Grocer with Orders		75		
		Weekly Allowance	2	00		
	6	Gift to Brother			1	00
	6	Balance			2	00
			5	20	5	20
	10	Cash on Hand	2	00		

DR. THOMAS GRIFFIN CR.

1916						1916					
Jan.	7	Buttons	13*	12	50	Jan.	10	By Cash	12*	10	00
	9	Lining	15	7	25		20	By Check	24	10	00
	15	Thread	21	1	25		20	By Mdse.	24	6	10
	17	Buttons	23	6	10			Balance		19	15
	24	Serge	29	18	15						
				45	25					45	25
Feb.	1	Balance		19	15						

*These figures show page in cash book from which these items were taken.

11. Inventories of (*a*) Household Goods and (*b*) Stock in a Store.

12. Post-office business.

Copies of pamphlets or circulars giving the mailing rates, the classes of mail, and money-order rates, should be obtained by the children. After these are read and understood by them a variety of examples may be solved, of which the following are illustrations. It is obvious that the tables of rates and "zones" should not be memorized.

(*a*) Find the cost of sending a letter weighing $5\frac{1}{2}$ oz. from New York to Boston.

(*b*) What would be the cost if the letter were registered? If sent by special delivery.

(*c*) A Sunday newspaper weighing $7\frac{1}{2}$ oz. is sent from your post office to San Francisco. What amount of postage is necessary?

(*d*) A package of shoes weighing 1 lb., 4 oz. is sent from your post office to Sandusky, Ohio. What will the cost be?

(*e*) What will it cost me to send my brother in St. Louis $125 by money order? Etc.

13. Notes.

Include "On demand," "Promissory note with interest."

14. Single entry bookkeeping.

Really nothing more than a coördination of cash account, personal (ledger) account, and previous information.

Suggestions for Teaching Business Forms.—1. *Use Actual Forms.*—As far as possible real forms should be used. This does not preclude the use of bills, invoices, notes, etc., ruled by the children. But in the main real bills, real checks, real deposit slips should be obtained from banks and business houses for this work. All firms do not use bills identical in form; all banks do not follow the same arrangement of items on checks. To bring real business forms into class prepares the child for any arrangement that he may meet later in life.

2. *Motorize or Dramatize Business Procedures in which These Forms are Necessary.*—Instead of calling upon children to imagine business conditions that make the use of these forms necessary, it is better to dramatize these situations. One child sends the other a bill; the latter compares it with his invoice and then writes his check. It is arranged to have the same child receive a number of checks which he indorses and enters upon a deposit slip preparatory to going to the bank. A sends B three bills and finally a monthly statement. B checks the monthly statement, discovers an error, and writes A, who acknowledges the overcharge and sends a new statement with proper credit items. The possibilities for this sort of dramatization are numerous and the opportunities for correlation with composition are very rich.

3. *Careful Gradation and Correlation through the Grades.*—Not all of one business form should be taught in any one grade, but it should be the aim of supervisors to distribute as many business forms as possible through the school course.

In an earlier grade a bill of one type is taught; in a later grade, another. In the fourth year the bill contains only whole numbers; in the fifth, fractions are used; in the seventh, double discounts and credit items are introduced. In this way the various business forms are reviewed, each repetition has a new point of view, and practice is given in the fundamental operations with whole numbers, fractions, decimals, and percentage.

4. *Lead Children to Apply the Work Personally.*—Much can be done to impress these business forms on the children by encouraging each one to keep a personal account and to assist his parents in making out bills, checks, receipts, statements, etc.

5. *Give Business Reasons for Arrangement of These Forms.*—Wherever possible, reasons should be given justifying the facts and the arrangement in these business forms. Children should know why checks and stubs are numbered, why one writes

"Twenty-three and 25/100.....Dollars,"

why there is a line between the amount and word "Dollars," why the form 25/100 is used, why the amount is written again on the check in figures, why one must begin writing the amount close to the left edge of the check—in a word, the conditions from which existing business forms and arrangements have gradually been evolved.

CHAPTER XVI

TEACHING THE ELEMENTS OF STATISTICS

What to Teach Children in Statistics.—In recent years statistics have assumed an important place in industry and commerce as well as in scientific research. It is evident that children must be given practice in reading, interpreting, and formulating statistical tables, averages, graphs, timetables, and numerical results of a similar nature. Although the work is of necessity elementary in its nature, nevertheless it can give the child a useful stock of facts and an insight into statistical forms and tabulations.

Averages.—In teaching children to compute averages, two types of examples must be differentiated. The first is illustrated by such examples as "A's weekly earnings are, 12, 14, 11, 12 and 9 dollars, respectfully, for each of 5 successive weeks. What is his average weekly wage?" Here the salaries are added, which gives \$58 for 5 weeks; the average is, therefore, \$58÷5 or \$11⅘ or \$11.60. Children are required to average their marks, knowing their ratings in each of six subjects; to average the daily attendance for the week, knowing the attendance each day, etc. In this form of computation there is little to explain, and nothing to confuse the class.

The second type involves reduction of different elements to a similar basis. Thus, all 6B children in a school were tested; their results are listed in the following column:

4 pupils have 50%	14 pupils have 75%
3 pupils have 55%	5 pupils have 80%
8 pupils have 60%	3 pupils have 85%
2 pupils have 65%	4 pupils have 90%
15 pupils have 70%	

What was the average for each pupil?

Here it is necessary to compute that 4 pupils scored 200 points, 3 pupils 165 points, etc. The results may be obtained in one of the two forms given below:

Usual Mode	*Suggested Mode*	
4 pupils scored 200 points	4 pupils scored 0 points over 50	000
3 pupils scored 165 points	3 pupils scored 5 points over 50	15
8 pupils scored 480 points	8 pupils scored 10 points over 50	80
2 pupils scored 130 points	2 pupils scored 15 points over 50	30
15 pupils scored 1050 points	15 pupils scored 20 points over 50	300
14 pupils scored 1050 points	14 pupils scored 25 points over 50	350
5 pupils scored 400 points	5 pupils scored 30 points over 50	150
3 pupils scored 255 points	3 pupils scored 35 points over 50	105
4 pupils scored 360 points	4 pupils scored 40 points over 50	160
58 pupils scored 4090 points	58 pupils scored ? points over 50	1190

$$\begin{array}{r} 70+ \\ 58\overline{)4090} \\ \underline{406} \\ 30 \end{array} = 70\% \qquad \begin{array}{r} 20+ \\ 58\overline{)1190} \\ \underline{116} \\ 30 \end{array}$$

58 pupils scored 1190 points over base
1 pupil averaged 20%+ over 50%
Average for each pupil=50%+20%+=70%+.

The second mode of solution is more rapid. All the statements and the number of words given above are not necessary in actual operation. They are given here to make clear the

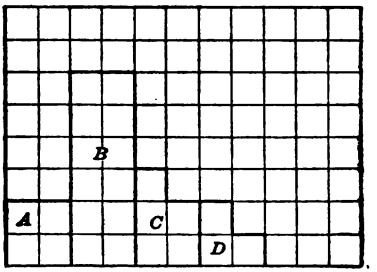
meaning and the suggested form of solution. The multiplications are simpler, the products to be added are smaller, and the results are usually obtained more rapidly, but with no less accuracy than in the first mode given above. The method of teaching averages will be discussed in the concluding part of this chapter.

Graphs and Diagrammatic Representations of Relative Quantities.—Graphs must be taught because they are so commonly used today in advertisements, newspaper articles, popular studies in magazines, and in propaganda literature. The modern geography usually expresses quantitive facts concerning population, areas, exports, imports, products, etc., by means of diagrams and graphs. Figures are not so vivid as lines and spaces, and mere statistics lack the power possessed by the graph for stirring imagination. For these reasons European schools have included a study of graphs in the elementary curriculum and progressive American textbook writers are giving them their well-deserved place in the course of study.

What Situations Shall Be Expressed in Terms of Graphs?—There is an endlesss variety of experiences that arises in the course of schoolwork that can well be expressed graphically. An incomplete summary of these will suggest a host of others to the reader: statistics from history and geography, class records in tests, individual records of progress in all or in some subjects, temperatures, athletic records, industrial activities of the community, civic problems dealing with population, number of voters, increase in tax rate, per capita cost, health statistics, attendance records, passengers carried by local transportation lines, solution of problems, etc. A few illustrations are offered.

1. *In Making Diagrammatic Representations.*

In a class of 48 children, 8 received A, 24 B, 10 C, and 6 D in proficiency. Represent the distribution of the proficiency of the class diagrammatically.



Each Box Represents 2 Children

A,	4	Boxes,	Represents	8	Children
B,	12	"	"	24	"
C,	5	"	"	10	"
D,	3	"	"	6	"

FIG. 32.

Upon examining a herd of cows it was found that rating the perfect cow 100 points, the cows were marked as follows: 10 at 90 points, 5 at 80, 15 at 75, 20 at 70, 20 at 65, 10 at 60, 15 at 55 and 15 at 50. Represent diagrammatically the distribution of the herd.

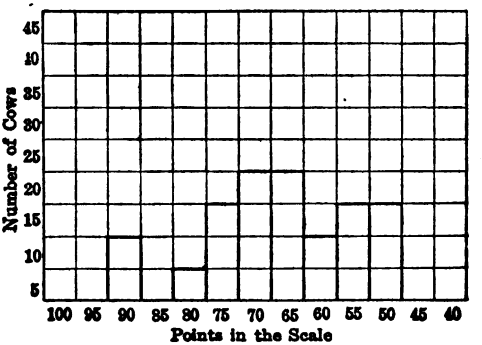


FIG. 33.

A school opened February, 1910, with 1702 pupils. Its attendance for the successive terms was as follows: 1721, 1770, 1810, 1840, 1865, 1870, 1865, 1868, respectively. Chart the increase or decrease of the school, term by term.

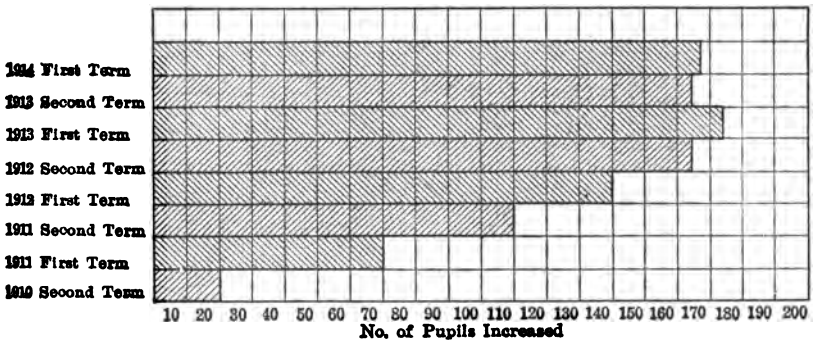


FIG. 34.

2. *Making Graphs of Single or Double Records.*

A trolley car makes its round trip on an average of once every hour. The number of passengers it carried each round trip, beginning at 1 A.M., was 40, 36, 30, 26, 21, 42, 80, 116, 92, 85, 81, 80, 124, 122, 104, 92, 80, 61, 52, 57, 68, 60, 61, 55, respectively. Make a graph showing the changes in the number of passengers hour by hour. (See Fig. 35.)

A pupil who failed in arithmetic the first month of the term made the following progress for the next 8 weeks, 58%, 62%, 65%, 70%, 72%, 72%, 75%, 73%. His classmate who passed the first month received the following ratings in the same 8 weeks: 65%, 62%, 62%, 60%, 62%, 55%, 60%, 64%. Compare the quality of work and the effort each made, by using graphs. (See Fig. 36.)

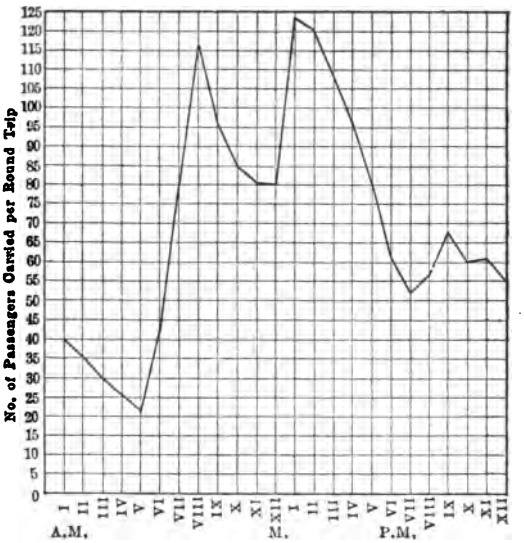


FIG. 35.

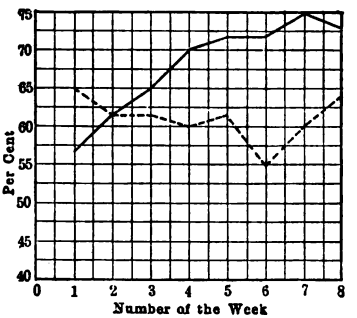


FIG. 36.

Heavy line represents first child. Dotted line represents second child.

3. *Interpreting Graphs.*

The following graph is presented to a class in order to give practice in interpreting the lines, reading the numbers, and in drawing tentative conclusions. Nothing in addition to the information contained below should be given to the children.

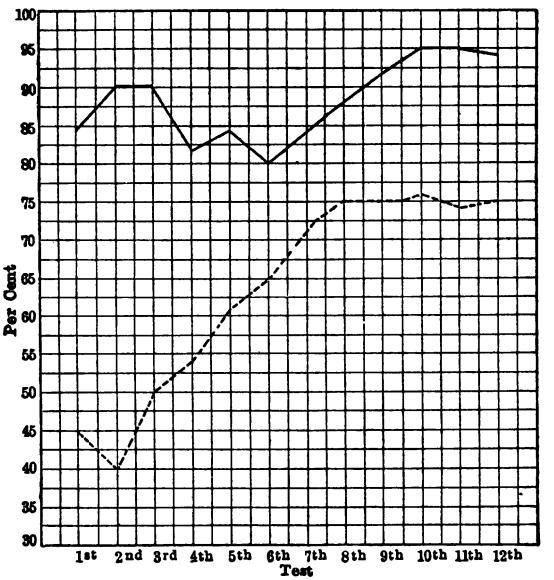


FIG. 37.

Graphs showing progress in arithmetic (heavy line) and grammar (dotted line) of a boy who liked arithmetic but thought he could never learn grammar.

After studying the two graphs children should state:

1. The ratings in each test in each subject.
2. The discouragement of the child in grammar at the beginning of the term.
3. The effort that this boy evidently made.
4. The progress in grammar.

5. The improvement in arithmetic as well as in grammar. Teachers will find children intensely interested in such exercises, especially when they are given in friendly rivalry to find who can "see" most in a graph.

4. *Solving Problems.*

By the aid of graphs, children can readily solve problems of the following type:

A train leaves New York at 2 P.M. and travels 30 miles per hour. At 4 P.M. another train leaves the same station at the rate of 40 miles per hour. At what time will the second train overtake the first?

The mode of solution of this example is illustrated in the accompanying graphs.

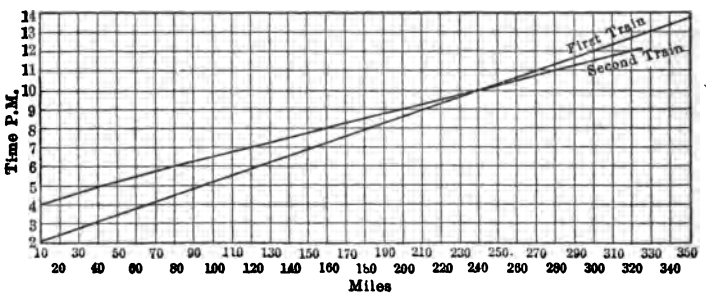


FIG. 38.

Second train overtakes first at 10 P.M. at a place 240 miles from New York, the starting station.

Timetables.—Practice should be given in reading timetables, in computing differences in distance and in time, probable cost at an average of two cents per mile, comparative cost of travel by water and by rail, etc. These timetables can usually be secured with great ease and such tables should be used as correlate with the grade work in geography. This is the best and most natural means of

impressing transportational facilities and commercial routes upon pupils.

Suggestions for Teaching Statistical Elements.—All these matters of averages, statistics, and graphs should be taught deductively, with as much explanation and justification for each practice as the children can comprehend. The aim should be to bring situations which are related to the studies or the interests of the children. Making a graph of the changing prices of commodities as they are announced day by day in the newspapers, or keeping a record, by a graph, of one's ratings in the important school subjects, is an illustration of the host of possibilities that give life and meaning to this branch of arithmetic. Throughout this work, these elements of statistics must be made a means to a higher end rather than an end in themselves—a means to a better understanding of the quantitative aspect of experience. If, as a result of this work, the children do not see more in the figures published daily concerning exports, products, cost of municipal and social enterprises, athletic records, etc., this study of the elements of statistics has been in vain.

An apparatus which is very useful in teaching graphs has been devised by Mr. Hugo Newman, the principal of the New York Training School for Teachers. A large board with properly ruled paper fastened on it is hung before the class. The facts to be represented on the horizontal and the vertical axes are written on the blackboard against which the wooden board rests or on cardboards easily put in place. Thumb tacks are put into the places that mark the changing direction of the graph. A string is stretched from tack to tack to represent the graph. When two or three graphs are to be plotted, thumb tacks and cord of two or three colors are used, each color representing a graph. This device saves time and enables the teacher to present the work on a scale large enough to be seen by every pupil in the room.

CHAPTER XVII

MEASURING THE RESULTS OF INSTRUCTION SCIENTIFICALLY

THE NEED OF SCIENTIFIC STANDARDS

The Threefold Standards Used to Measure Educational Results.—If we accept any of the conceptions of education which have gained currency—"social efficiency," "preparation for complete living," "adjustment to the complete environment," "complete self-realization"—we perceive at once that the process of education is the process of life. Nevertheless vague impressions and personal opinions, denominational faith or changing prejudices, have dictated its procedure and evaluated its results. Efficiency of education is measured by (1) personal opinion, (2) the traditional examination, and (3) fixed units of a definite scale. The first two methods are the time-honored devices of supervisors and teachers; the third is now commanding attention.

Limitations of Personal Opinion as a Standard.—Personal opinion as a standard of judgment is valuable in direct proportion to the quality of one's native judgment, the character of one's training, and the extent and nature of one's experience. Prejudice, inaccuracies of observation and inference, self-interest—these are constantly coloring all educational measurement based on personal opinion. Nevertheless this method of judgment is to-day the prevailing one. A new psychology stresses the overwhelming importance of the muscular sense in mental development and concludes

that spelling should be taught through the muscle. At once an order is promulgated from on high that children must not visualize their spelling words; every pupil is now seen writing his words on scraps of paper, on the desk with his finger, or in the air. A pedagogical prophet arises, who asserts, in terms psychological, that penmanship drills for arm movement, posture, and coördination violate proper procedure. To learn to write, children must write. Busy supervisors, overwhelmed by the sycophantic array of reasons, overthrow the old system and impose a new one that develops habits of incorrect arm movement and injurious bodily posture. "Every number is a ratio," a new school of philosophical educators asserts, and ratio is applied without stint or mercy until personal opinion dislodges this pedagogical fetish to enthrone a new one.

Limitations of the Traditional Examination.—The method of testing by means of the traditional examination with its multiplicity of questions is another prevailing procedure. These forms of measuring efficiency proceed on the assumption that the prevailing curricula and methods of teaching and of supervision are correct. Educational efficiency is still determined by the fund of information that the child carries away. In the final analysis, these examinations gauge progress by what is temporary. Facts known today are forgotten tomorrow. They test for what is accidental, for a class that has, by mere chance, reviewed a portion of a given subject before an examination scores higher ratings than another which has received better instruction. They test for what is relatively unimportant, for not facts, but mental habits, mental fiber, are the ultimate desiderata of instruction. No system of testing and comparing should be tolerated unless it gauges the worth of a student by what is permanent, what is vital, and of final worth.

The Nature of the New Scientific Standards.—The third means of measuring efficiency is by a scientific scale, evolved

after careful experimentation and scientific summation. We must have an accurate yardstick in terms of which we can tell how much concentration a child should normally develop after five years of work in arithmetic; what degree of accuracy in tables should be acquired in six years; or what should be the degree of proficiency in reasoning, after seven years of learning arithmetic. We must know what degree of speed and legibility children should normally attain after four years of class instruction in penmanship. In the same way we must have standards for measuring successive progress, year after year, in drawing, in spelling, and in composition. Otherwise we fall back on the errors of personal opinion. Until these scientific standards are evolved, we have no means of coming to a conclusion worthy of respect, on some very vital problems in school work.

Problems to be Answered by Scientific Tests.—The need of these scientific standards can be seen from our lack of knowledge of problems of vital concern in the education of our children. Among these we may mention the following:

1. What degree of excellence in composition shall we expect in each year of the high school? Some children in the first year of the high school write better compositions than those written by third-year students, while others produce work inferior to that of an undergraduate of the elementary school. What degree of proficiency shall we set for each year? In each high-school year children are taught principles of composition. What standard does the second-year teacher seek to attain which the first-year teacher cannot? The writer has asked teachers this question, but aside from a vague impression that they expect a higher degree of proficiency, they had no definite way of making the high-school course in composition progressive, not repetitive.

2. What is the relation between the time spent in a given subject and the degree of proficiency attained by children?

We have no answer except a vague guess. *A priori*, proficiency in a subject will be in direct proportion to the time spent on it. But investigations thus far do not warrant such a conclusion. Time expenditure is no guaranty of proficiency. We were told in an earlier chapter that if we were to select at random a school from a list of schools that give more than average time to arithmetic, the chances are about equal that we would have chosen a school where the work is below standard, and, conversely, if we were to select a school from among those that devote less than average time to this subject, we would probably find the children doing superior work in it.

3. To what extent shall we rationalize in arithmetic? Shall children know the " why " for every step in the multiplication by numbers of two or three orders or shall they habituate the process? The advocates of rationalization are as insistent as their opponents, yet neither side has any more definite basis than personal opinion.

4. Shall children be taught to subtract by a method of addition or by a method of subtraction? The former method is now in the ascendency and is the prescribed method in the New York City schools because it has a long series of theoretical advantages. Have we shown experimentally that children taught by one method do speedier and more accurate work than do those taught by another?

5. How shall we grade our pupils in a given class? Some would put all children of similar ages in one class; others would put children of related abilities in the same class. Some would have children stay one year in each class, others only a half year, while a third faction would have each year divided into four terms, and thus make for greater frequency of promotion. Whose is the infallible judgment that has evolved the true answer?

6. What are the factors governing fatigue in children? Will pupils taught arithmetic at 10:30 in the morning do

worse work than children taught at 9:30? If children leave the classroom at 11 A. M. and return at 1:30, after having had shop work, play period, and an hour for lunch, are they in an unfavorable condition for the study of arithmetic? The opponents of one system of school management are as positive that arithmetic taught in the afternoon will not produce good results as the proponents are that the quality of the work will in no way deteriorate. Careful physical examinations and an accurate observation of results under experimental conditions will do more towards effecting a solution of the question of fatigue than a continuation of the present scholastic debate in which nothing more is offered than the opinions of the disputants.

New York City is spending over forty million dollars annually for the education of its youth. Has the city a single mode of measuring a single phase of the product of its educational system? Can we conceive of an industry involving an annual investment of half that sum continuing the even tenor of its way without stopping to take stock and to compare cost and product?

These problems show us the futility of the sentimental plea that we must not apply concrete and scientific standards of measurement to instruction, for who can measure the worth of personal contact with the teacher, the value of a new attitude toward life, or the benefit of mental discipline? No parent applies such a sentimental standard to determine the worth of a private teacher who instructs his child in music or foreign languages. Why should we become ultra-sentimental in judging class instruction?

THE DEVELOPMENT OF SCIENTIFIC STANDARDS FOR MEASURING EFFICIENCY OF INSTRUCTION IN ARITHMETIC

It is now our purpose to take a bird's-eye view of what has been accomplished in the attempt to evolve scientific standards for measuring the efficiency of instruction in

arithmetic. Since the chronological treatment shows the progress from simple beginning to scientific standards, we shall examine the method and the results of each succeeding worker in this field of pedagogical research.

The Work of Rice.—Rice is the pioneer in endeavors to formulate standards in arithmetic. In 1902 he gave his test questions to about 6000 children of grades 4A through 8B in 17 schools representing 7 cities. The test questions are given in the Appendix. The most striking features in the results are the low ratings and the wide range of capabilities of children in the same grade. In the seventh year, the averages for pupils varied from 8.9 per cent to 81.1 per cent; in the eighth year, from 11.3 per cent to 91.7 per cent. The work of marking was carefully planned, each example being marked (1) correct in principle and work; or (2) correct in principle only; or (3) incorrect.

Rice then set himself the task of interpreting the results in a constructive sense. He takes as his basis of judgment that the probable factors determining successful work are (1) home, (2) the size of the class, (3) age of pupils, (4) time of day when a subject is taught, (5) time devoted to the subject, (6) home study, (7) methods of teaching, (8) teaching ability of the staff of instructors, (9) the course of study, and (10) supervision of teachers and standards for measuring results. Each of these factors is scrutinized in the light of the children's results.

1. The home is shown to be negligible in determining quality of results. The odds were against the schools in "good" neighborhoods, where there were "aristocratic" homes in which the child's time was its own and conditions were favorable to study. 2. The size of the class, that source of concern in modern school management, was also shown to be insignificant as a determining force. Schools with the lowest averages had very small classes and some of the schools with the best results had very large classes.

3. The age of the pupil must be considered, Rice concludes, because the older pupils grasp the work more readily and the results are correspondingly better. The following table shows the grades, average age in each grade, the rating in per cent attained by younger and older children, the quality of work in the highest and the lowest school. A comparison of the figures bears out the investigator's conclusions. 4. The schools examined in the morning did

Grade	4		5		6	
	Average Age	Per Cent	Average Age	Per Cent	Average Age	Per Cent
Six Highest Schools....	11.9	62.8	12.6	84.3	13.4	96.3
Six Lowest Schools.....	11.0	29.0	12.0	49.8	13.4	61.4

Grade	6		7		8	
	Average Age	Per Cent	Average Age	Per Cent	Average Age	Per Cent
Five Highest Schools...	13.4	49.5	14.1	71.9	14.11	90.4
Six Lowest Schools.....	13.4	11.0	13.11	29.0	14.5	38.0

better than those tested in the afternoon, but the difference was slight, barely 3 per cent. 5. Those schools that gave most time to the subject were not correspondingly high in attainment, for the school that ranked next to the lowest gave the greatest amount of time daily to arithmetic; the school that averaged 80 per cent and the one that scored only 25 per cent gave the same number of minutes daily. 6. The amount of homework given to the pupils seemed to bear no direct relation to the proficiency attained, and Rice concludes that homework puts an added burden on the children, saps their energies, and has practically no compensating results. 7, 8 and 9. The investigator's conclusions, that teachers whose work is modernized, who have teaching gifts and professional skill, and who follow a course of

study that is organized on social standards, do not affect the results obtained, are indeed difficult to accept. One is made to feel that a previous conclusion is being forced upon him and he is, therefore, left unconvinced.

10. Rice's final conclusions are that proper supervision of teachers and definitely standardized measures of results are the determining factors in making for efficient work in arithmetic. These standards of attainment do not permit each teacher to be the judge of his or her own work, nor each principal to be the sole judge of the standing of his school or the worth of his teachers. Standard tests set up definite, well-crystallized expectations that give direction to each teacher's efforts. Rice does not see, however, that these standards may become fixed and formal; that the routine test encourages routine work and discourages initiative. His work was too limited, and the absolutely uniform conditions, necessary for determining a standard, were not secured in each school tested; but his contribution is significant because of the impetus it gave the scientific movement in arithmetic.

The Work of Stone.—Dr. C. W. Stone set himself the task of securing data on (1) the nature of the product of the first six years in arithmetic, and (2) the relation of "distinctive procedure in arithmetic to resulting abilities." With these ends in view he organized a test of two parts. Part A, in fundamental operations, and Part B, in reasoning problems. Part A sought to combine all the difficulties in the fundamental operations. Part B sought to test the reasoning power of children in problems that deal with situations equally familiar to all, that are well graded, and that omit all unduly large numbers, all "catch" elements, and memory requirements like "320 rods = 1 mile." The tests are given in an Appendix.

The technique of the test was carefully worked out. Part A, consisting of fundamentals, was allowed 12 min-

utes and was designed to keep the most rapid workers very busy. This time limit permits all children to complete the first six examples and yet prevents any child from being without work. Part B, consisting of reasoning problems, was allowed 15 minutes, a period determined by actual test on 100 children. Here, too, the time limit permits all to do six or seven problems but none to complete the test. The mode of scoring was developed by giving an arbitrary value of 1 to each step in the fundamental operations. Thus, example 5, Part A, was assigned 4 credits because there are 2 multiplication processes and 2 addition processes thus:

$$\begin{array}{rl}
768 & \\
604 & \\
\hline
2972 & \text{1 credit} \\
4608 & \text{1 credit} \\
\hline
\underbrace{463772}_{\substack{\text{no addition,}\\ \text{no credit,}\\ \text{2 additions,}\\ \text{2 credits,}\\ \text{no addition,}\\ \text{no credit,}}} &
\end{array}$$

By studying the results of 100 children, it was determined to give the problems in Part B (reasoning) values of 1 for the first, 1.4 for the sixth, 1.6 for the eighth, 2 for the ninth, etc. These tests were given in 1907 in 26 school systems, in 152 classes, to 6000 children representing the East and the Middle West. It is seen, therefore, that the work was given with due regard to uniformity and scientific precision.

The most important conclusions summed up by Stone as a result of his investigations are listed below;

1. Marked discrepancies exist among children of the same grade who are grouped on the basis of uniform teaching and approximately equal attainment.

2. Ability in one branch of arithmetic is no guaranty of

approximately equal ability in another, even though the two abilities are both in fundamental operations. There is no such thing as a single or unified or combined "ability in arithmetic." What we call ability in arithmetic is a composite of abilities—a complex of abilities.

3. The averages set up in the study are the results obtained by testing 6000 children and are offered as norms by means of which any school system can be measured.

4. There seems to exist no relation between time cost and proficiency developed in arithmetic.

5. As time cost is increased, up to certain limits, children seem to do better work in fundamental operations than in reasoning problems. The probable conclusions are:

(*a*) Time given to fundamental operations is more productive of results, or,

(*b*) More of the total time of arithmetic should go to fundamental operations, or,

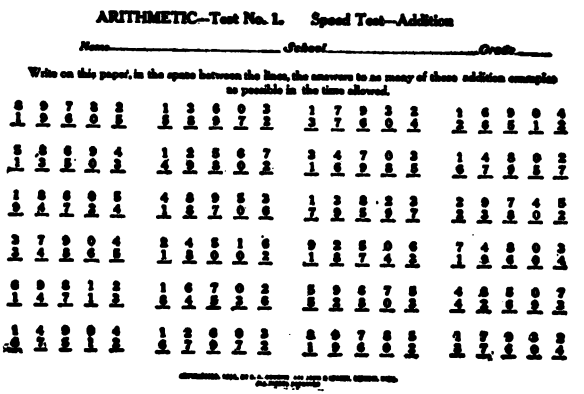
(*c*) Power to reason is less conditioned by amount of teaching than is skill in fundamental operations.

6. There seems to exist no relation between the character of the course of study and children's abilities. Stone submitted the 26 courses of study of those school systems whose children he tested, to 21 competent judges who rated these courses in order of rank. The composite judgment then showed Course x the best, Course y, the next, etc. When the children's ratings were compared with the course of study, no relationship could be discerned. The chances are about even that children taught from a course of study rated as low will attain higher scores, and conversely.

Stone's work showed clearly the possibilities of developing scientific standards for measuring the instructional results in arithmetic and the need of submitting time-honored opinions of methods of teaching and of courses of study to careful scrutiny, and to the acid test of unbiased collective judgment.

The Courtis Tests.—*Aim of These Tests.*—The most scientific standards yet devised for the teaching of arithmetic are those formulated by S. A. Courtis. These Courtis tests are not examinations in the ordinary sense of the term; they do not test for the knowledge of facts taught, but for the power and the habits that result from the work accomplished. For this reason the same test is given to all children in all grades under the same conditions. In all measurement processes the measuring medium must be constant. So, too, in measuring intellectual power, the reasoning standard must remain unchanged.

Organization of the Tests.—The Courtis tests are divided into eight parts, each seeking to gauge one specific ability. Thus: Test 1 determines mastery in addition combinations; Test 2, in subtraction combinations; Test 3, in multiplication combinations; Test 4, in division combinations. Test 5 measures the child's motor ability; Test 6, power for speed reasoning; Test 7, ability in fundamental operations; Test 8, power to reason in two-step problems.

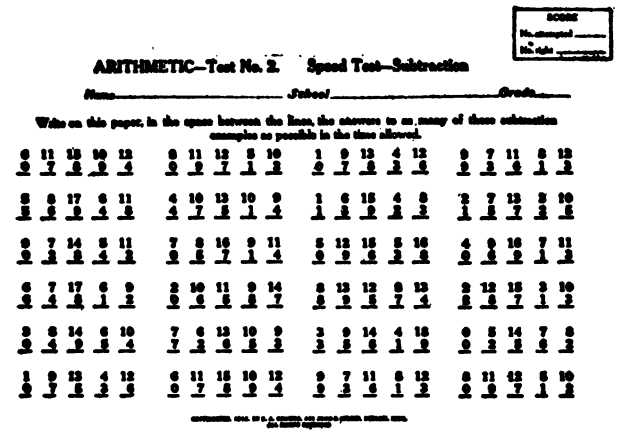

ARITHMETIC—Test No. 1. Speed Test—Addition

Name _____ School _____ Grade _____

Write on this paper, in the space between the lines, the answers to as many of these addition examples as possible in the time allowed.

FIG. 39.

SCORE
No. attempted _____
No. right _____



ARITHMETIC—Test No. 2. Speed Test—Subtraction

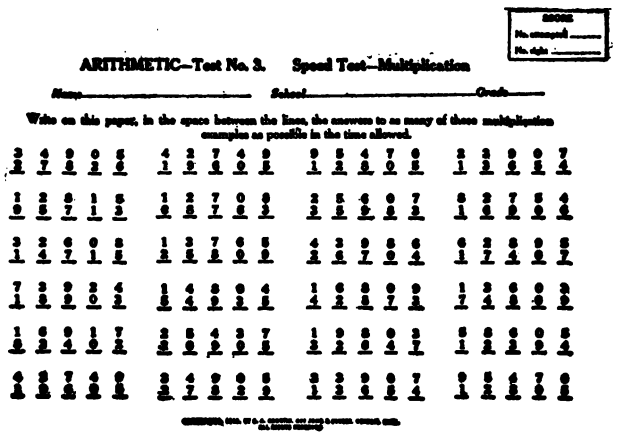
*Name*_____ *School*_____ *Grade*_____

Write on this paper, in the space between the lines, the answers to as many of these subtraction examples as possible in the time allowed.

6	11	15	10	12	8	11	12	5	10	1	9	13	4	12	9	7	11	8	13
0	7	8	9	4	0	9	7	1	2	0	7	8	3	6	9	3	6	1	3
5	8	17	6	11	4	10	13	10	9	1	6	15	4	8	2	7	13	3	10
5	6	9	4	8	4	7	5	1	4	1	3	9	2	3	1	5	7	2	5
9	7	14	5	11	7	8	16	9	11	5	12	15	5	16	4	9	16	7	11
0	2	8	4	2	0	5	7	1	4	0	9	6	3	8	0	6	9	1	3
6	7	17	6	9	2	10	11	9	14	8	13	12	8	13	2	12	15	3	10
6	4	8	1	2	0	6	5	8	7	8	9	5	7	4	2	8	7	1	3
3	8	14	6	10	7	6	13	10	9	3	9	14	4	18	0	5	14	7	8
0	4	9	5	4	7	2	6	5	3	3	5	6	1	9	0	2	5	6	2
1	9	13	4	12	6	11	15	10	12	9	7	11	8	12	8	11	12	5	10
0	7	5	3	6	0	7	8	9	4	9	3	6	1	3	0	9	7	1	2

FIG. 40.

SCORE
No. attempted _____
No. right _____



ARITHMETIC—Test No. 3. Speed Test—Multiplication

*Name*_____ *School*_____ *Grade*_____

Write on this paper, in the space between the lines, the answers to as many of these multiplication examples as possible in the time allowed.

3	4	9	0	5	4	2	7	4	9	9	5	4	7	0	2	3	9	0	7
2	7	8	3	6	1	9	6	0	5	1	2	8	0	5	1	3	6	5	4
1	2	8	1	5	1	2	7	0	8	2	5	6	0	7	8	2	7	5	4
0	5	7	1	3	0	5	7	0	3	3	5	9	8	3	1	6	9	0	6
3	2	6	0	8	1	3	7	6	5	4	3	9	8	6	6	2	8	9	5
1	4	7	1	5	2	5	8	0	9	2	6	7	0	4	1	7	4	0	7
7	3	9	2	4	1	4	8	0	4	1	6	8	0	9	1	3	6	0	3
1	8	9	0	3	5	4	9	3	5	4	2	8	7	3	7	4	8	0	9
1	6	9	1	7	2	5	4	3	7	1	9	8	0	3	5	8	6	0	5
8	3	4	0	2	3	0	9	0	5	3	2	6	4	7	1	2	3	9	4
4	3	7	4	9	3	4	9	0	5	3	3	9	0	7	9	5	4	7	0
5	9	6	0	8	2	7	8	2	9	1	3	6	5	4	1	2	5	9	5

FIG. 41.

Arithmetic Test No. 4. Speed Test. Division. Score No. Attempted.. No. right.....

Name.....*School*.....*Grade*.....

Write on this paper, in the space between the lines, the answers to as many of these division examples as possible in time allowed.

1)8 5)30 8)72 1)0 9)36 2)6 4)24 7)63 6)0 8)32 9)9 3)21 6)48 1)1 5)10 3)9 4)32 6)36 2)0 7)28

5)5 4)36 9)54 8)0 4)12 1)5 2)16 8)48 1)2 9)27 1)4 5)35 9)45 2)2 3)12 8)8 4)28 5)40 2)2 8)16

5)0 3)24 9)63 2)4 8)24 6)6 3)27 8)64 1)2 4)16 3)6 4)20 7)49 1)3 2)8 1)7 2)10 7)42 1)1 6)18

4)4 3)15 9)81 7)0 6)12 4)4 6)30 8)56 1)0 7)14 7)7 2)18 6)42 3)0 7)21 5)5 2)14 8)40 9)0 5)15

1)6 7)35 6)54 1)3 5)20 1)9 5)25 7)56 3)3 9)18 1)5 3)18 9)72 4)0 6)24 1)4 2)12 5)45 3)3 4)8

3)9 4)32 6)36 2)0 7)28 2)6 4)24 7)35 6)0 8)32 1)8 5)30 8)72 1)6 9)36 9)9 3)21 6)48 1)1 5)10

FIG. 42.

Arithmetic. Test No. 5. Speed Test. Copying Figures Score No. attempted.. No. right.. ....

Name.....*School*.....*Grade*..

Copy on this paper, in the space between the lines, as many of the printed figures as possible in the time allowed. Write as rapidly as possible, but form the figures as carefully as in working examples.

24967 42976 62947 72964 24976 42967 62974 72946

26974 46927 64972 74926 26947 46972 64927 74962

27946 47932 67924 76942 27964 47926 67942 76924

24967 42976 62947 72964 24976 42967 62974 72946

26974 46927 64972 74926 26947 46972 64927 74962

24967 42976 62947 72964 24976 42967 62974 72946

FIG. 43.

Arithmetic. Test No. 6. Speed Test—Reasoning. Score No. attempted.. No. right.....

Name.....*School*.....*Grade*....

Do not work the following examples. Read each example through, make up your mind what operation you would use if you were going to work it, then write the name of the operation selected in the blank space after the example. Use the following abbreviations: "Add." for addition, "Sub." for subtraction, "Mul." for multiplication, and "Div." for division.

	Operation	
1. The children of a school gave a sleigh-ride party. There were 9 sleighs used, and each sleigh held 30 children. How many children were there in the party?.....		
2. Two school-girls played a number game. The score of the girl that lost was 57 points and she was beaten by 16 points. What was the score of the girl that won?.....		
3. A girl counted the automobiles that passed a school. The total was 60 in two hours. If the girl saw 27 pass the first hour how many did she see the second?.....		
4. On a playground there were five equal groups of children each playing a different game. If there were 75 children altogether, how many were there in each group?.		
5. A teacher weighed all the children in a certain grade. One girl weighed 70 pounds. Her older sister was 49 pounds heavier. How many pounds did the sister weigh?		
6. A club of boys sent their treasurer to a store to buy baseballs. If they expected him to buy 7 balls at 45 cents each, how much money did they give him to spend?		
7. A girl walked to school and back twice a day. If the total number of blocks she walked each day was 48, how many blocks from the school did the girl live?.....		
8. One day in vacation a boy went on a fishing trip. During the day he caught 27 fishes. If he caught 12 of them in the morning, how many did he get in the afternoon?...		
9. A girl was five times as strong as her small sister. If the stronger girl was able to lift a weight of 100 pounds, how large a weight could the smaller girl lift?.....		
10. Two boys' houses, 26 blocks apart, were on opposite sides of a school. If one house was 15 blocks east of the school, how many blocks west was the other house?.....		

FIG. 44.

Arithmetic. Test No. 6. Speed Test—Continued

	Operation	
11. Five boys played marbles until at the close of a game each boy had the same number. If one of the boys then had nine marbles, how many were there altogether?.....		
12. Thirteen children entered a certain eighth grade during the year. If there were 43 children in the grade in September, how many were there in the grade by June?		
13. A large box of colored chalk held 144 pieces. If a sixth-grade teacher used 36 pieces in the course of a year, how many pieces were left for the next year?.....		
14. In a certain school there were 400 children. If each room had seats for fifty children, and if all the places were taken, how many rooms were there in the school?.....		
15. A girl brought a collection of 37 postal cards to school one day and her friends there gave her 19 cards more. How many cards did she then have to take home?.....		
16. Five boys gathered nuts which they put into one large pile. Out of this they made five small piles of 197 nuts each. How many nuts were there in the large pile?		

FIG. 44—Continued.

Just as the distance from 7 to 8 on a ruler is the same as from 5 to 6, so, too, in the Courtis tests every point in a test is equally difficult. If a question is twice as difficult as the preceding one, it is given twice the credit. No element is determined by personal opinion, but is tested with large numbers of children. A child who secures a score of 40 is hence twice as high as the child who scored only 20.

In all the Courtis tests more work is given than can be done in the time set. Work thus put on a speed basis makes the results more nearly uniform as we go from child to child and grade to grade. The results disclose at once weaknesses wherever they exist. "Speed is not insisted on for the sake of speed, but because speed best ministers to uniformity of conditions and to expose weaknesses."

Arithmetic. Test No. 7. Fundamentals

Score
No. attempted..
No. right.....

Name.....*School*.....*Grade*....

In the blank space below, work as many of these examples as possible in the time allowed. Work them in order as numbered, writing each answer in the "answer" column before commencing a new example. Do no work on any other paper.

No.	Operation	Example	Answer	Rt.	
1	Addition	*a* 25+830+122=..(Write answer in this column)			
		b 232+8021+703+3030=.....			
2	Subtraction	*a* 5496−163=.....			
		b 943276−812102=.....			
3	Multiplication	2012×213=.....			
4	Division	158664÷132=.....			
5	Addition	6134+213+4800+6005+3050+474=.....			
6	Subtraction	73210142−49676378=.....			
7 8	Multiplication	46508×456=.....			
9	Division	27217182÷6=.....			
10 11	Division	3127102÷463=.....			
12 13	Addition	85586+69685+39397+95836+37768+69666+78888+54987=.....			
14	Subtraction	15655431−5878675=.....			
15 16	Multiplication	78965×678=.....			
17	Division	44502486÷7=.....			
18 19	Division	5373003÷769=.....			

FIG. 45.

Arithmetic. Test No. 8. Reasoning. Score No. attempted.. No. right.....

Name.....*School*.....*Grade*....

In the blank space below, work as many of the following examples as possible in the time allowed. Work them in order as numbered, entering each answer in the "answer" column before commencing a new example. Do not work on any other paper.

	Answer	
1. A party of children went from a school to a woods to gather nuts. The number found was but 205 so they bought 1955 nuts more from a farmer. The nuts were shared equally by the children and each received 45. How many children were there in the party?.....		
2. One summer a farmer hired 43 boys to work in an apple orchard. There were 35 trees loaded with fruit and in 57 minutes each boy had picked 49 apples. If in the beginning the total number of apples on the trees was 19,677, how many were there still to be picked?.....		
3. A girl found by careful counting that there were 87 letters more on a page of her history than on a page of her reader. In the first 29 days of school she read 31 pages in each book. How many more letters each day did she read in one book than in the other?.....		
4. The children of a school made small boxes to be filled with candy and given as presents at a school party. 600 were needed. In 4 days grades 3 to 7 made 20, 25, 83, 150, and 150 boxes. The eighth grade agreed to make the rest. How many did the eighth grade make?		
5. A girl's record in spelling for 5 days was 19, 18, 20, 16, and 20 words spelled correctly out of 20. If each of the 16 children in the grade had had the same record, what would have been the total number of words spelled correctly by that grade in 5 days?.....		
6. A party of boys went on a long bicycle trip. They traveled 1702 miles in 37 days. A number of men then joined the party, and soon the party was traveling 58 miles per day. How much change in the number of miles ridden a day did the presence of the men make?.....		

FIG. 46.

Arithmetic. Test No. 8. Reasoning—Continued

	Answer	
7. A teacher corrected 2400 arithmetic test papers, 2295 of these he marked "poor," "fair," "good," etc. All the others were marked "unsatisfactory." If each of the papers in this group had 47 mistakes, what was the total number of mistakes in the unsatisfactory papers?.....		
8. In two schools five teachers recorded the number of blocks the children walked in going to and from school. The total for one school was 3000 blocks; for the other 2400. The number of children in both schools was 216. How many blocks did each child walk a day?.....		
Total		

FIG. 46—Continued.

Let us see how carefully each test is constructed. Test 1, the addition test, contains all the 45 combinations that can be made with any two digits taken additively, the reverse of each, as well as the zero combinations. These 45 combinations are then listed under 5 orders of difficulty.

Group A, "the very easy"—	0	0	0	0
	1	2	3	4
	–	–	–	–
Group B, "the easy"—	1	1	2	3
	2	3	1	1
	–	–	–	–
Group C, "the average"—	2	2	3	4
	3	4	2	2
	–	–	–	–
Group D, "the hard"—	2	2	3	
	8	9	8	
	–	–	–	
Group E, "the very hard"—	5	5	5	
	6	7	8	
	–	–	–	

But this gradation is no mere guesswork by teacher or supervisor. The combinations are taught under experimental conditions in different schools and in different cities and careful records are made of *time* and *effort* required to find the answer to a given combination. A careful tabulation of the data gives an accurate classification. In Test 1, there are horizontal groups of 5 combinations; each group is made up of an equal number of the " very easy," " easy," " average," " hard " and " very hard " combinations. The result is an absolutely uniform scale. The same principle of organization obtains in Tests 2, 3 and 4.

The object of Test 5 is to measure motor ability rather than the ability to copy figures. Motor ability is often a most helpful aid in analyzing the causes of a child's apparent failure. Let us assume that child A scores half the class average in the subtraction test, but is normal in work in Test 5; child B also scores half the class average in the subtraction test, and also scores only half the class average in Test 5. It is evident, therefore, that the poor score of child B in the subtraction test may be due to limited motor ability and not to a lack of tables, whereas child A's work shows clearly the need of further drill in subtraction tables. " Other things being equal, the individual who is able to expend greater energy at a greater rate has the greater ability."[1]

Test 5 shows the same painstaking care and accuracy of organization. Uniformity in the scale is secured by a repetition of the same figures and by a careful gradation of these figures according to the ease of formation. Experimentally we find that the average man can make three 1's while he makes one 5; that 2, 4, 6, 7 take the same time for writing; that 9 requires the longest time for formation. Each horizontal group of five figures has one figure of

[1] Courtis, Interim Report on New York City Investigation. P. 19.

each grade of difficulty and a 9 in the center of the group, thereby establishing a rhythm that tends toward uniformity.

Test 6 is a speed reasoning test. Uniformity is secured by (1) repeating each of the four operations four times; (2) asking such problems as require only one operation; (3) so wording each problem that it takes up the same space; (4) selecting a non-technical content for each problem out of the everyday experience or reading of the child. But despite all efforts the absolute uniformity is absent because (1) the inherent difficulty in the four operations varies; (2) not all children have an equally good apperceiving stock of English words and many children, laboring under the limitation of paucity of interpretational vocabulary, may be hampered in their work; (3) the varying phraseology makes for varying degrees of comprehensibility. Further changes must be made in this test to produce greater uniformity.

Test 7 seeks to measure: (1) knowledge and control of the process in four fundamental operations; (2) knowledge of basic combinations; (3) ability to "borrow" and "carry"; (4) ability to copy figures; (5) ability to follow instructions, which is so necessary in commercial work. To secure uniformity in this test—that is, to make every problem equal in difficulty to every other—each example was analyzed into its minutest process of (1) copying, (2) multiplying in thought, (3) actual paper multiplication, (4) carrying a figure, (5) copying the answer, etc. Each process was then experimentally accorded its allotted worth in a scale of points, e.g., 1, 2, 5 or 10, depending on time it consumed and the effort to work it out. Each example was reduced to 50 points, while those that came to 100 points were given a double count. The examples may, therefore, be regarded as advancing units on an absolute scale.

Test 8 has all the limitations of Test 6 but in more exaggerated form. The language element is a more serious consideration here than in the earlier test. Each example has two operations and a pair of examples includes all four operations. Each example was analyzed as explained in Test 7 and each constituent process was evaluated until each example was reduced to 50 points. This test aims, therefore, to present an absolute scale of eight equal subdivisions. Experience will undoubtedly modify this test, for we do not know whether a combination of addition and multiplication is as difficult as one of multiplication and division, or whether the introduction of unnecessary numbers gives a more reliable index of reasoning ability.

Sequence of Tests.—These tests are given in two groups. Group A includes Tests 5, 2, 8 and 3, while Group B contains Tests 4, 6, 1, 7. Test 5 is given first because it is least determined by study and prepares children for the conduct of these tests. Test 8 is given early so that it may not be influenced by others. Test 7 is given last so that it will be influenced by the others. Test 2 precedes Test 8 because it is a speed test and least likely to influence 8, Test 1 is given almost last, because if given first it tends to prompt children to revert to addition in all other tests. The sequence of these tests is no accident, but the result of careful observation under experimental conditions.

Time Allowed for Each Test.—The time allotted to each of these tests may prove interesting if not amazing; Tests 1, 2, 3, 4, 5 and 6 are given in one minute each, Test 7 in twelve minutes, and Test 8 in six minutes. All the tests, including time for explanations, passing of papers, etc., can be given in sixty to eighty minutes.

Scoring the Tests.—The scoring of these papers is governed by a procedure that is carefully designed to shut out all personal judgment and that is the result of wide experience and experimentation. Each person who does the

scoring must take the tests himself and must familiarize himself with the details of a special manual. One illustration of these conditions may exemplify the scientific character of the scoring. Beyond the fourth year there is no scoring of "rights" in Tests 1 to 4, inclusive. The number of examples attempted is the final score. After a study of about 100,000 scores, it was found that "mistakes are too few in number to pay for the labor expended. The sight of two figures in position for addition seems to act as a stimulus to which the habitual response must be made. A child who does not know his tables is apparently unable to make up and write any answer. . . . So powerful is the grip of habit that most children persist until the correct answer is recalled, ignorance of the tables being shown, not by incorrect answers but by low scores" (of attempts).

Series B of the Courtis Tests.—Supplementary Courtis tests, known as Series B, have recently been formulated. These seek to measure power in (1) addition, (2) subtraction, (3) multiplication and (4) division in actual examples rather than the mere mastery of combinations in these processes. The examples involve the factors necessary for a perfect mastery of each of the four fundamental operations. These tests are given from the third year up, once or twice each term, and the records are sent on with the children, so that the supervisor may determine when and under what circumstances children have acquired most of their habits in the basic arithmetical operations. The first test is allowed eight minutes; the second, four; the third, six, and the fourth, eight. The same scientific precision in the technique of tests, i.e., organization, scoring, uniformity of condition, etc., that obtains in Series A is characteristic of these supplementary standards. These tests, together with tentative standard scores, are given in the Appendix.

STANDARDS OF MEASUREMENT AS AIDS IN CONSTRUCTIVE SUPERVISION AND ADMINISTRATION

Educational standards of measurement serve a variety of uses in the administration of schools and in the supervision of instruction. We shall attempt to suggest their possibilities.

1. Standards Test the Gradation and Classification of Pupils.—Educational standards have already proved their worth in detecting the wide range of abilities among children in the same class. The Courtis Tests have shown very convincingly that in any one grade there are more children off the average than on it; that the number securing an average result is very small; that in each grade there is a greater divergence of ability than similarity; that 80 per cent of the children of a grade can be replaced by children of other grades without changing the total scores; that 35 per cent of the children in a fourth-year class do as well as some in the eighth year; that 81 per cent of the children in a sixth-year class do as well as some in a seventh-year class; that a business man who employs a school graduate of average arithmetical ability secures the services of a youth whose work can be equaled by 5 per cent of the fourth-grade pupils. These tests disclose the fact that for 52 schools and 905 classes 45 per cent of the grade membership of any grade will exceed the average score of that grade; 35 per cent of the grade membership will exceed the average of the next higher grade; 35 per cent of the grade membership will fall below the average of the grade below it.

2. Standards Detect Waste in Class Teaching.—The statistics cited to show the lack of gradation in the classification of pupils serve also to indicate the amount of waste entailed in teaching a class as a class. In our schools a given proficiency is attained by moving the class as a whole. But the lack of a basis of uniform needs or capacities in the organization of a class means that in all class teaching some

pupils are overstrained while others are enervated mentally because the rate of progress is so ill adjusted to their possibilities. Children markedly different in attainments are subjected to the same methods of teaching and the same degree of drill. Such teaching results in waste for pupils at both ends of the scale of proficiency and is stultifying instead of invigorating, repressive instead of expressive.

3. Standards Supply Each Grade with Tentative Norms to Measure Progress.—After an accepted standard has been applied to a large number of children and the ratings properly averaged, there results a series of tentative norms in each subject for each grade:

STANDARD SCORES

Test No.	No. 1	No. 2	Nos. 3 and 4	No. 5	No. 6		No. 7		No. 8	
					Atts.	Rts.	Atts.	Rts.	Atts.	Rts.
Grade 3.....	26	19	16	58	2.7	2.1	5.0	2.7	2.0	1.1
Grade 4.....	34	25	23	72	3.7	3.0	7.0	3.3	2.6	1.7
Grade 5.....	42	31	30	86	4.8	4.0	9.0	4.9	3.1	2.2
Grade 6.....	50	38	37	99	5.8	5.0	11.0	6.6	3.7	2.8
Grade 7.....	58	44	44	110	6.8	6.0	13.0	8.3	4.2	3.4
Grade 8.....	63	49	49	117	7.8	7.0	14.4	10.0	4.8	4.0
Grade 9.....	65	50	50	120	8.6	7.8	15.0	11.0	5.0	4.3
Time Allowances, minutes.....	1	1	1	1	1	1	1	12	6	6

In this table we see that by the Courtis tests the standard score for a child in the fifth year is as follows: In addition, 42; in subtraction, 31; in copying figures, 86, etc. A child who is up to or beyond grade average in any one of the component abilities of arithmetic may be excused from further drill in that phase of the subject and be given opportunity to invest his time in subjects in which he is weak. These tentative grade norms, when translated graphically, give us what are known as " comparative graphs."

An analysis of this chart may prove helpful; it can be summed up readily in the following points (see Fig. 47):

1. Each type or test of ability has its own horizontal line.
2. The grades or years are represented by vertical divisions.
3. Each test is graduated according to possible scores.
4. The norms for any grade are so arranged as to fall one under the other.
5. The complete standard score, or complete norm, for any grade is a vertical line.

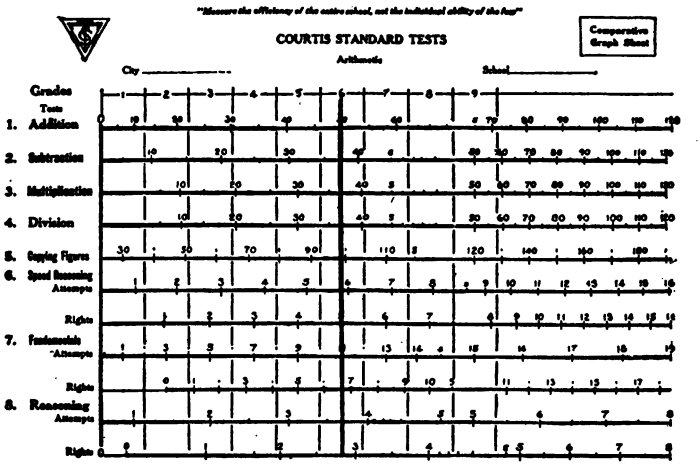


FIG. 47.

COMPARATIVE GRAPH: showing complete standard score for a sixth-year pupil.

4. Standards as Aids in Diagnosing Individual Cases.—Tentative norms can be made the means of diagnosing individual cases and of deciding on proper remedial exercises. An analysis of the accompanying comparative graph shows us a sixth-year child above grade in addition, subtraction, division combinations, well beyond the norm in fundamental operations, but below in speed reasoning and in reasoning problems. It is obvious that this child needs no further drill in abstract problems; its arithmetic should emphasize thought problems rather than mere exercises in addition, subtraction, multiplication, etc. The next graph gives us a

diagnosis of five children: Child A needs further drill in addition and multiplication tables, and practice to increase motor ability; in the other powers this child is up to the grade norm. Child B is behind in every ability and needs more drill in all phases of arithmetic. He is obviously misgraded. Child C, although behind in tables, needs no further stimulation, for his progress in Tests 7 and 8 (Fundamentals and Reasoning), indicates unmistakable ability. Child D has evidently been overdrilled in abstract combinations and needs practice in application to examples as well as in reasoning problems.

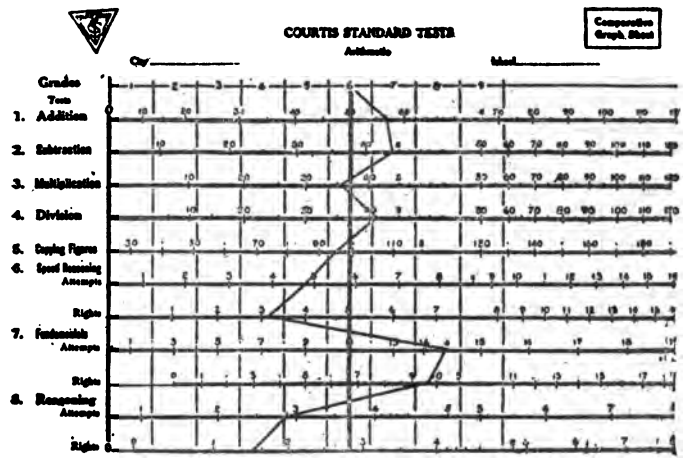


FIG. 48.

INSTRUCTIONS.

Find on the proper scale for each test the point corresponding to the grade average or individual score for that test, and join point to point. The curve of any grade in a "good" school or of a "good" scholar in the grade will be wholly within the boundaries for that grade. Excellence is shown by balance as well as by high scores.

The scales are based upon "Standard Scores" derived from the measurement of nine thousand children in sixty different schools in ten different states.

5. Standards as Aids in Judging Merits of Courses of Study.—In current educational practice there is no objective means of determining the relative worth of contending

courses of study. The advocates of each are emphatic in their praise of some one and bitter in their denunciation of all the others. Personal opinion usually prevails upon personal opinion and a compromise course is selected. Standards of measuring educational products give us a means of gauging the worth of a course of study. On the assumption that the following graphs represent the work of an entire school system, year by year, we conclude that abstract

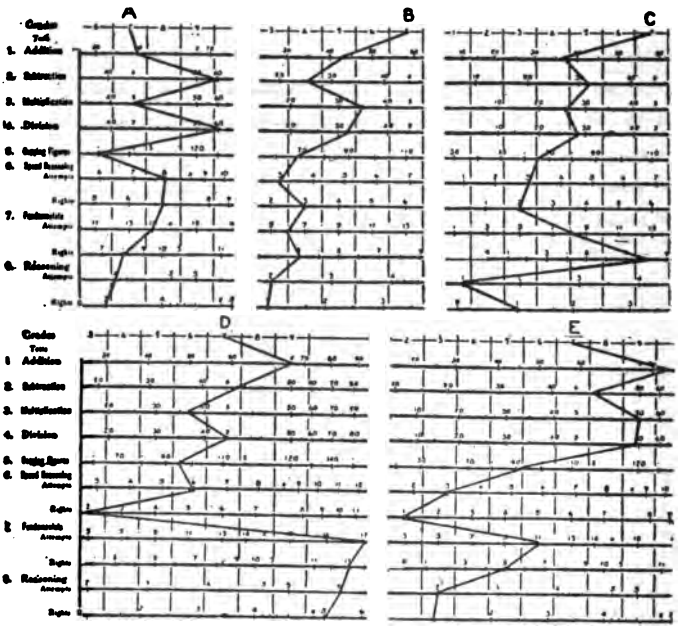


FIG. 49.

work is emphasized at the cost of reasoning power. Every year studied shows a uniform advance of the norm in combinations and in fundamentals, but a score below the norm in reasoning work. Dr. Courtis gives in Fig. 51, a comparison of two grades, one, a typical New York City seventh-

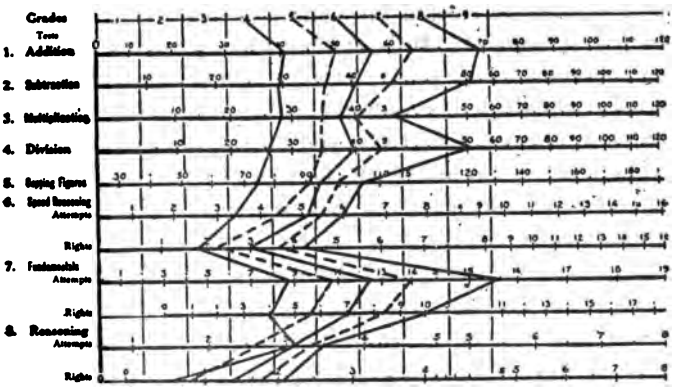


FIG. 50.

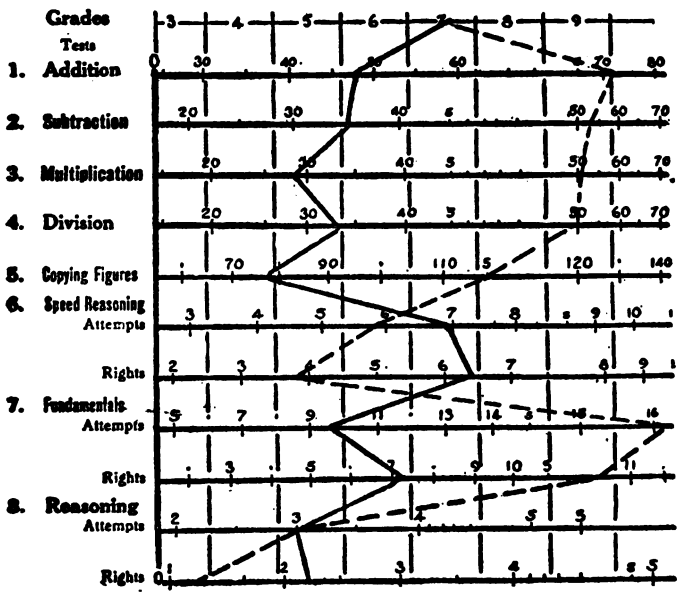

Grades
Tests
1. Addition
2. Subtraction
3. Multiplication
4. Division
5. Copying Figures
6. Speed Reasoning
Attempts
Rights
7. Fundamentals
Attempts
Rights
8. Reasoning
Attempts
Rights

FIG. 51.

year product, the dotted line, and a second a similar grade in a private school. It is evident that we have here two extreme courses of study. Such results, obtained with properly constituted standards, give a scientific basis for changes in courses of study and in methods of teaching.

These are by no means all the uses of standards of measuring instructional efficiency. This general survey of possibilities shows clearly that further methodology in arithmetic, as in other subjects, must be evolved after careful experimentation and investigation under scientific conditions. Not until the method of the laboratory is applied to the problems of teaching will pedagogy be purged of its sterile twaddle and personal opinions, and attain the dignity worthy of a science of teaching.

APPENDIX I

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APPENDIX II

STANDARD TESTS

The Rice Tests

FOURTH YEAR

1. A man bought a lot of land for $1743, and built upon it a house costing $5482. He sold them both for $10,000. How much money did he make?

2. If a boy pays $2.83 for a hundred papers, and sells them at four cents apiece, how much money does he make?

3. If there were 4839 classrooms in New York City, and 47 children in each classroom, how many children would there be in the New York schools?

4. A man bought a farm for $16,575, paying $85 an acre. How many acres were there in the farm?

5. What will 24 quarts of cream cost at $1.20 a gallon?

6. A lady bought 4 pounds of coffee at 27 cents a pound, 16 pounds of flour at 4 cents a pound, 15 pounds of sugar at 6 cents a pound, and a basket of peaches for 95 cents. She handed the storekeeper a $10 note. How much change did she receive?

7. I have $9786. How much more must I have in order to be able to pay for a farm worth $17,225?

8. If I buy 8 dozen pencils at 37 cents a dozen, and sell them at 5 cents apiece, how much money do I make?

FIFTH YEAR

1. A man bought a lot of land for $1743, and built upon it a house costing $5482. He sold them both together for $10,000. How much did he make?

2. If a boy pays $2.83 for a hundred papers, and sells them at four cents apiece, how much does he make?

3. What will 24 quarts of cream cost at $1.20 a gallon?

4. If I buy 8 dozen pencils at 37 cents a dozen, and sell them at 5 cents apiece, how much money do I make?

5. A flour merchant bought 1437 barrels of flour at $7 a barrel. He sold 900 of these barrels at $9 a barrel, and the remainder at $6 a barrel. How much did he make?

6. How many feet long is a telegraph wire extending from New York to New Haven, a distance of 74 miles? There are 5280 feet in a mile.

7. A merchant bought 15 pieces of cloth, each containing 62 yards. He sold 234 yards. How many dress patterns of 12 yards each did he have left?

8. Frank had $3.08. He spent $\frac{1}{4}$ of it for a cap, $\frac{1}{7}$ of it for a ball, and with the remainder bought a book. How much did the book cost?

SIXTH YEAR

1. If a boy pays $2.83 for a hundred papers, and sells them at 4 cents apiece, how much does he make?

2. What will 24 quarts of cream cost at $1.20 a gallon?

3. If I buy 8 dozen pencils at 37 cents a dozen, and sell them at 5 cents apiece, how much do I make?

4. A flour merchant bought 1437 barrels of flour at $7 a barrel. He sold 900 of these barrels at $9 a barrel, and the remainder at $6 a barrel. How much did he make?

5. If a train runs $31\frac{2}{3}$ miles an hour, how long will it take the train to run from Buffalo to Omaha, a distance of 1045 miles?

6. If a map 10 inches wide and 16 inches long is made on a scale of 50 miles to the inch, what is the area in square miles that the map represents?

7. The salt water which was obtained from the bottom of a mine of rock salt contained .008 of its weight of pure salt. What weight of salt water was it necessary to evaporate in order to obtain 3896 pounds of salt?

8. A gentleman gave away $\frac{1}{7}$ of the books in his library, lent $\frac{1}{3}$ of the remainder, and sold $\frac{1}{5}$ of what was left. He then had 420 books remaining. How many had he at first?

SEVENTH YEAR

1. If a map 10 inches wide and 16 inches long is made on a scale of 50 miles to the inch, what is the area in square miles that the map represents?

2. The salt water which was obtained from the bottom of a mine of rock salt contained 0.08 of its weight of pure salt. What weight of salt water was it necessary to evaporate in order to obtain 3896 pounds of salt?

3. A gentleman gave away $\frac{1}{7}$ of the books in his library, lent $\frac{1}{8}$ of the remainder, and sold $\frac{1}{5}$ of what was left. He then had 420 books remaining. How many had he at first?

4. A farmer's wife bought 2.75 yards of table linen at \$0.87 a yard, and 16 yards of flannel at \$0.55 a yard. She paid in butter at \$0.27 a pound. How many pounds of butter was she obliged to give?

5. If coffee sold at 33 cents a pound gives a profit of 10 per cent, what per cent of profit would there be if it were sold at 36 cents a pound?

6. Sold steel at \$27.60 a ton, with a profit of 15 per cent, and a total profit of \$184.50. What quantity was sold?

7. If a woman can weave 1 inch of rag carpet a yard wide in 4 minutes, how many hours will she be obliged to work in order to weave the carpet for a room 24 feet long and 24 feet wide? No deduction is to be made for waste.

8. A fruit dealer bought 300 apples at the rate of 5 for a cent, and 300 at 4 for a cent. He sold them all at the rate of 8 for 5 cents. What per cent did he gain on his investment?

EIGHTH YEAR

1. If a map 10 inches wide and 16 inches long is made on a scale of 50 miles to the inch, what is the area in square miles that the map represents?

2. The salt water which was obtained from the bottom of a mine of rock salt contained 0.08 of its weight of pure salt. What weight of salt water was it necessary to evaporate in order to obtain 3896 pounds of salt?

3. A gentleman gave away $\frac{1}{7}$ of the books in his library, lent $\frac{1}{8}$ of the remainder, and sold $\frac{1}{5}$ of what was left. He then had 420 books remaining. How many had he at first?

4. A man sold 50 horses at $126.00 each. On one-half of them he made 20 per cent, and on the other half he lost 10 per cent. How much did he gain?

5. Sold steel at $27.60 a ton, with a profit of 15 per cent, and a total profit of $184.50. What quantity was sold?

6. A fruit dealer bought 300 apples at the rate of 5 for a cent, and 300 at 4 for a cent. He sold them all at the rate of 8 for 5 cents. What per cent did he gain on his investment?

7. The insurance on $\frac{3}{4}$ of the value of a hotel and furniture cost $420.00. The rate being 70 cents on $100.00, what was the value of the property?

8. Gunpowder is composed of niter 15 parts, charcoal 3 parts, and sulphur 2 parts. How much of each in 360 pounds of powder?

Stone's Test

In Reasoning

The main purpose of the reasoning test is the determination of the ability of 6A children to reason in arithmetic. To this end, the problems as selected and arranged are meant to embody the following conditions:

1. Situations equally concrete to all 6A children.
2. Graduated difficulties
   (*a*) As to arithmetical thinking,
   (*b*) As to familiarity with the situation presented.
3. The omission of
   (*a*) Large numbers,
   (*b*) Particular memory requirement,
   (*c*) Catch problems,
   (*d*) All subject matter except whole numbers, fractions, and United States money.

The test is purposely so long that only very rarely did any pupil fully complete it in the fifteen-minute limit.

The Tests as Given

The following are reproductions of the tests. They were printed separately and each pupil was furnished with a copy.

Work as many of these problems as you have time for; work them in order as numbered.

1. Add

2375
4052
6354
260
5041
1543

2. Multiply 3265 by 20.
3. Divide 3328 by 64.
4. Add

596
428
94
75
302
645
984
897

5. Multiply 768 by 604.
6. Divide 1918962 by 543.
7. Add

4695
872
7948
6786
567
858
9447
7499

8. Multiply 976 by 87.
9. Divide 2782542 by 679.
10. Multiply 5489 by 9876.
11. Divide 5099941 by 749.
12. Multiply 876 by 79.
13. Divide 62693256 by 859.
14. Multiply 96879 by 896.

Solve as many of the following problems as you have time for; work them in order as numbered:

1. If you buy 2 tablets at 7 cents each and a book for 65 cents, how much change should you receive from a two-dollar bill?

2. John sold 4 *Saturday Evening Posts* at 5 cents each. He kept $\frac{1}{2}$ the money and with the other $\frac{1}{2}$ he bought Sunday papers at 2 cents each. How many did he buy?

3. If James had 4 times as much money as George, he would have $16. How much money has George?

4. How many pencils can you buy for 50 cents at the rate of 2 for 5 cents?

5. The uniforms for a baseball nine cost $2.50 each. The shoes cost $2 a pair. What was the total cost of uniforms and shoes for the nine?

6. In the schools of a certain city there are 2200 pupils; $\frac{1}{2}$ are in the primary grades, $\frac{1}{4}$ in the grammar grades, $\frac{1}{8}$ in the high school, and the rest in the night school. How many pupils are there in the night school?

7. If $3\frac{1}{2}$ tons of coal cost $21, what will $5\frac{1}{2}$ tons cost?

8. A newsdealer bought some magazines for $1. He sold them for $1.20, gaining 5 cents on each magazine. How many magazines were there?

9. A girl spent $\frac{1}{3}$ of her money for carfare, and three times as much for clothes. Half of what she had left was 80 cents. How much money did she have at first?

10. Two girls receive $2.10 for making buttonholes. One makes, 42, the other 28. How shall they divide the money?

11. Mr. Brown paid one-third of the cost of a building; Mr. Johnson paid $\frac{1}{2}$ the cost. Mr. Johnson received $500 more annual rent than Mr. Brown. How much did each receive?

12. A freight train left Albany for New York at 6 o'clock. An express left on the same track at 8 o'clock. It went at the rate of 40 miles an hour. At what time of day will it overtake the freight train if the freight train stops after it has gone 56 miles?

COURTIS TESTS IN FUNDAMENTAL OPERATIONS

SERIES B

Arithmetic Test No. 1. Addition.

Score
No. Attempted.
No. Right.....

Form 2

You will be given eight minutes to find the answers to as many of these addition examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

127	996	237	386	186	474	877	537
375	320	949	463	775	787	845	685
953	778	486	827	684	591	981	452
333	886	987	240	260	106	693	904
325	913	354	616	372	869	184	511
911	164	600	261	846	451	772	988
554	897	744	755	595	336	749	559
167	972	195	833	254	820	256	127
554	119	234	959	137	533	258	323
237	564	632	674	421	258	326	267
492	278	263	158	988	885	770	854
679	947	318	745	465	600	753	684
513	522	949	121	114	874	199	358
468	989	746	437	676	726	469	938
731	243	653	426	729	142	643	333
856	334	428	953	235	355	698	493
302	669	456	674	190	947	186	775
925	142	532	329	406	351	173	239
873	622	485	172	236	537	648	584
168	479	871	426	578	227	396	157
332	283	524	951	877	725	389	617
419	791	919	537	916	598	374	624
934	808	722	989	543	906	859	467
493	253	456	565	593	763	191	369
529	419	216	230	956	195	423	511
156	952	862	673	439	480	849	245
224	522	424	258	309	102	342	233

Name.....Grade.....

Arithmetic. Test No. 2. Subtraction

Score
No. Attempted....
No. Right.....

Form 1

You will be given four minutes to find the answers to as many of these subtraction examples as possible. Write the answers on this paper directly underneath the examples. You are not expected to be able to do them all. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

107795491	75088824	91500053	87939983
77197029	57406394	19901563	72207316
160620971	51274387	117359208	47222970
80361837	25842708	36955523	17504943
115364741	67298125	92057352	113380936
80195261	29346861	42689037	42556840
64547329	121961783	109514632	125778972
48813139	90492726	81268615	30393060
92971900	104339409	60472960	119811864
62207032	74835938	50196521	34379846
137769153	144694835	123822790	80836465
70176835	74199225	40568814	49178036

Name.....Grade.....

Arithmetic. Test No. 3. Multiplication

Score
No. Attempted....
No. Right.....

Form 1

You will be given six minutes to work as many of these multiplication examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

8246	3597	5739	2648	9537
29	73	85	46	92
4268	7593	6428	8563	2947
37	640	58	207	63
5368	4792	7942	3586	9742
95	84	72	36	59
6385	8736	5942	6837	4952
48	502	39	680	47
3876	9245	7368	2594	6495
93	86	74	25	19

Name.....Grade.....

Arithmetic. Test. No. 4. Division

Score
No. Attempted....
No. Right.....

Form 1

You will be given eight minutes to work as many of these division examples as possible. You are not expected to be able to do them all. Do your work directly on this paper; use no other. You will be marked for both speed and accuracy, but it is more important to have your answers right than to try a great many examples.

25)6775	94)85352	37)9990	86)80066
73)58765	49)31409	68)43520	52)44252
37)14467	86)60372	94)67774	25)9750
68)39508	49)28420	52)21112	73)33653
28)23548	54)48708	39)32760	67)61707
45)33795	76)57000	93)28458	82)29602

Name.....Grade.....

TENTATIVE SCORES. SERIES B. JUNE, 1914.

Grades	Addition		Subtraction		Multiplication		Division	
	Attempts	Correct	Attempts	Correct	Attempts	Correct	Attempts	Correct
3.....	4.0	2.0	4.0	1.0				
4.....	6.0	3.0	6.0	3.0	4.5	1.5	3.5	1.0
5.....	7.5	4.0	8.0	5.5	7.0	4.0	5.0	3.0
6.....	9.0	5.0	10.0	7.0	8.5	5.5	6.5	5.0
7.....	10.5	6.5	11.5	8.5	10.0	6.5	8.5	7.0
8.....	12.0	8.0	12.5	10.0	11.5	8.0	10.5	9.0

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